

SL(2, \mathbb{C}) and SU(2) Connection Variable Formulations of Kerr Isolated Horizon Geometries for Loop Quantum Gravity

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A construction of both self-dual SL(2, \mathbb{C}) and SU(2) connection variable formulations for the description of the degrees of freedom of classical, rotating Kerr isolated horizon geometries is presented. These descriptions are based on sets of connection Hamiltonian variables instead of the spacetime metric. The analysis is motivated in a concrete, physical manner based on the stationary, axisymmetric Kerr solution of the vacuum Einstein equations, evaluated in a proper, well-defined frame of reference, on which isolated horizon boundary conditions are imposed. Having derived the kinematical part of such an isolated horizon phase space setting, one can set up a conserved presymplectic structure for the study of dynamical aspects of black hole theory. Since black holes play a crucial role in various fields like quantum gravity, mathematical physics, astrophysics and cosmology, or numerical relativity, one has to deal with different models describing these objects. The quasi-local framework studied in this paper is appropriate for covering most of the physical settings involving black hole dynamics. Moreover, the SU(2) connection variable formulation of classical Kerr isolated horizons allows directly for a semiclassical treatment of rotating quantum black holes in the context of loop quantum gravity.

I. INTRODUCTION

A. Classical Black Holes

In a lecture in 1967, John Archibald Wheeler introduced the term "black hole" as a part of spacetime from which nothing can escape, not even light. An infalling observer that has entered the scope of this particular region, is trapped forever and cannot send any information to another observer stationed outside since the inward pull of the gravitational field becomes so strong that the escape velocity would be greater than the vacuum speed of light. The primary formation process of these parts of spacetime is gravitational collapse, e.g., of a heavy object such as a star that, during the collapse, is squeezed into a pathological state covering an infinitesimally small region of spacetime with infinite curvature and density. After the extremely dynamical phase of the collapse, when the system can be considered, for the time being, as unperturbed or as isolated, it is abating into a simple stationary state, in which all the information defining the inner structure, the topology and dynamical aspects of the former, uncollapsed object seem to be irrecoverable. One is left with just three extensive observables for the characterization of the black hole: the mass M , the angular momentum J and the charge Q .

In more mathematical terms, the classical standard (global) representation of black holes is given by the 3-parameter Kerr-Newman family of highly symmetric vacuum solutions of the Einstein field equations of general relativity (Misner, Thorne & Wheeler, 1973). These solutions possess curvature singularities which cannot be resolved by a simple change of the underlying coordinate system, i.e., they describe geometrical regions, where the curvature of spacetime, in the framework of this model, actually becomes infinitely large. There, the domain of validity of the Einstein field equations of gravity comes to an end, hinting toward the incompleteness of general relativity. The Kerr-Newman solutions admit Cauchy horizon boundaries, called event horizons, of the black hole region that surround the singularity. The outer event horizon exhibits some characteristics of diodes since observers can only pass inward through the horizon toward the singularity of the black hole and never conversely. Usually, describing black hole geometries, an exterior observer can only refer to the outer event horizon boundary of the black hole region, because when stationed in the outside spacetime bulk, one is completely cut off from any inside information since the outer event horizon is a trapped surface, i.e., all light cones at the horizon point toward the compact interior region. Hence, the horizon structure (and not the curvature singularity), as the apparent defining feature of a black hole, is all that can be quantified and allows for mathematical manageability.

B. Non-Classical Black Holes

As long as classical dynamical changes, e.g., due to the accretion of surrounding matter, or quantum effects are disregarded, black holes are especially simple gravitational systems. However, considering classical, local dynamics or the microphysics of black holes (for the fundamental understanding of the origin of black hole entropy and of how to resolve curvature singularities), black holes become extremely complicated and one needs a more sophisticated description. For this to succeed, one has to deal with serious problems stemming from the global nature of the canonical definition of the event horizon structure (Wald, 1984), because measurements of black hole observables can only be addressed after having constructed the entire spacetime, which makes this definition inadequate for a description of any local physics. Therefore, the globality aspect creates profound intricacies when one wants to deal with black hole dynamics in classical and quantum gravity. Calculations carried out at a semiclassical level (Bekenstein, 1973; Hawking 1975), where the geometries are treated classically while matter fields are quantized, indicate that generic black holes in their final equilibrium stationary states at late times radiate as perfect black bodies at Hawking temperature $T_H = \hbar/(8\pi k_B M)$ and have an entropy $S_{BH} = k_B A/(4l_p^2)$, where M is the mass of the black hole and A is its total event horizon surface area, and thus, seem to slowly evaporate over time. Note that various forms of Hawking radiation (not only the thermal component) have to be considered for black holes with residual masses corresponding to thermal energies that, at least, do reach the rest masses of the lightest, massive elementary particles. In this semiclassical setting, serious problems arise with respect to the dynamical radiation process combined with the definition of a black hole horizon based on a global spacetime boundary. In order to avoid ill-posedness or unphysical systems, one has to revise the black hole horizon concept in the context of local models of semiclassical and full quantum gravity, respectively. Hence, one has to study geometrical boundary structures that are more concrete and quasi-local than event horizons. Recent attempts to address this problem are known under the names of isolated and dynamical horizons (Ashtekar & Krishnan, 2004). Isolated (equilibrium) and dynamical (non-equilibrium) horizon geometries are well-defined, classical, complete quasi-local frameworks for physical black holes, that are commonly used in loop quantum gravity, fully resolving the problematic issue of globality in dynamical situations. They cover all essential, local features of black hole event horizons and allow step-by-step evolutions of black holes. Thus, the knowledge of the full spacetime is no longer required. Accordingly, there is no information on the outside geometrical bulk structures encoded in the isolated and dynamical horizons, which means that additional physical inputs are needed. An obvious example of such an input comes from the aforementioned Hawking effect. The Hawking temperature has to be included into the horizon model by hand since the local horizon geometry does not know anything about the states of the scalar fields, which are the origin of the thermal properties of the black hole, existing outside in the bulk.

C. Black Holes in Loop Quantum Gravity

The present paper is concerned with the classical, rotating Kerr-type of the equilibrium isolated horizon models, which is one specific symmetric rotator amongst the many rotating isolated horizon geometries. In particular, the aim is to construct certain gauge formulations of these structures in terms of self-dual $SL(2, \mathbb{C})$ and $SU(2)$ connection variables, referred to as the Ashtekar and the Ashtekar-Barbero connection variables, providing a firm basis for quantization in the framework of loop quantum gravity. The original idea behind these connection variable formulations is the following. Using a set of configuration variables that foliate spacetime into a family of space-like surfaces, which are labeled by a time coordinate (3 + 1-split), general relativity can be cast in a Hamiltonian (canonical) representation (Arnowitt, Deser & Misner, 1959, 1960). According to this description, the Einstein equations are given by very complicated constraint equations (Gauss and diffeomorphism constraints for the kinematics and Hamiltonian constraint for the dynamics). These constraint equations can be drastically simplified by means of $SL(2, \mathbb{C})$ or $SU(2)$ connection variables (Ashtekar, 1986), reformulating them in the shape of the Yang-Mills constraint equations of particle physics. In this particular phase space framework of Hamiltonian general relativity, the kinematics is expressed in terms of both a flux variable, which is a densitized triad (a function depending on the metric tensor), and the holonomy of either the 3-dimensional $SL(2, \mathbb{C})$ or the $SU(2)$ gauge connection, serving as independent configuration variables. Geometric gravity in the constrained $SL(2, \mathbb{C})$ and $SU(2)$ gauge representations allowed for the rapid development of non-perturbative, background-independent quantum gravity and is a good starting point for the analysis of quantum isolated horizon geometries. Describing rotating Kerr isolated horizons in terms of $SU(2)$ connection variables opens up the possibility of a semiclassical, quantum-geometrical treatment in the context of loop quantum gravity (Frodden et al., 2012), by first constructing a conserved presymplectic structure, capturing the horizon boundary degrees of freedom by a $SU(2)$ Chern-Simons theory to account for the dynamics in the classical phase space, with the help of the isolated horizon Ashtekar-Barbero curvature (the principal result of this paper), followed by a quantization procedure by means of the geometric area operator and its eigenvalue spectrum defined on the kinematical loop quantum gravity Hilbert space of spin network states (Barbero G., Lewandowski & Villaseñor, 2012; Engle et al., 2010; Engle & Liko, 2013; Kaul, 2012; Perez & Pranzetti, 2011).

Here, the $SL(2, \mathbb{C})$ and the $SU(2)$ connection variable formulations of rotating Kerr isolated horizons are derived in a descriptive and physically intuitive manner, by exploiting the well-known classical Kerr vacuum solutions of general relativity. The article is organized as follows. In Section II, a short overview of the general definition and of the characteristics of isolated horizon boundaries is given. Section III contains an explicit step-by-step prescription of how to construct Kerr isolated horizon geometries. The complex, self-dual Ashtekar and the real Ashtekar-Barbero connection variable formulations of rotating Kerr isolated horizons, respectively, are introduced in Section IV. A procedure of how to realize a black hole quantum theory in the framework of loop quantum gravity, originating from the $SU(2)$ connection variable Hamiltonian representation, is explained in Section V. Finally, in Section VI, the results are summarized. The appendices contain proofs regarding the fulfillment of the isolated horizon boundary conditions and a discussion of the case of slowly rotating Kerr isolated horizon geometries.

II. OVERVIEW OF ISOLATED HORIZON GEOMETRIES

In standard classical general relativity, black holes, seen by exterior observers, are usually described in terms of event horizons. The definition of an event horizon as the future (past) boundary of the causal past (future) of future (past) null infinity is global, and, therefore, it is inappropriate for many real-world physical scenarios involving dynamical aspects like accretion and emission processes or to properly account for the local horizon degrees of freedom for a microscopic description of black hole entropy, because one is only able to raise questions concerning physical black hole observables after having constructed and, thus, possessing the full information on the entire spacetime, which makes event horizons by definition inadequate for conducting any local, dynamical studies. Hence, if one were to properly analyze for example the supermassive black holes in the centers of active galactic nuclei, one would be forced to deal with more local structures than event horizons, which can be realized with the notions of isolated and dynamical horizons. While the intricate dynamical horizons capture quasi-local, non-equilibrium situations in black hole evolution, with non-vanishing net fluxes emerging from accretion and emission processes, isolated horizons are used for the modeling of the much simpler quasi-local equilibrium cases (for extensive reviews on the topic, see, e.g., Ashtekar & Krishnan (2004) and Diaz-Polo & Pranzetti (2011)). Note that, in general, isolated horizons allow the occurrence of matter and radiation distributions in the spacetime bulk, but prohibit their interactions with the black hole.

In this paper, the focus is on isolated horizons, which can be classified into three different types with respect to their symmetry group. In physical terms this means that they are either non-rotating (type I), rotating around an internal symmetry axis (type II), or distortedly rotating (type III). Since the basic non-rotating case has already been elaborately analyzed (Ashtekar et al., 1998; Ashtekar, Corichi & Krasnov, 2000; Ashtekar, Fairhurst & Krishnan, 2000; Ashtekar, Beetle & Lewandowski, 2002; Engle et al., 2010; Sahlmann, 2011), here, one is concerned with the rotating type II isolated horizons (Ashtekar, Engle & Van Den Broeck, 2005). In particular, a discussion of rotating Kerr isolated horizons and their formulation in $SL(2, \mathbb{C})$ and $SU(2)$ connection variables is given. Before proceeding any further with the mathematical construction of the Kerr black hole isolated horizon models, a descriptive, general definition of isolated horizons is presented.

Isolated horizons in classical general relativistic physics can account for equilibrium states of black holes in a quasi-local framework. Denoting the isolated horizon with Δ , it constitutes a sub-manifold of a spacetime $(\mathcal{M}, g_{\mu\nu})$ with the following characteristics. Δ is a null surface, which is foliated by a particular family of marginally trapped, or rather isolated, 2-spheres S^2 in such a way that certain geometrical quantities, which are defined intrinsic to Δ , are time-independent. It has the topology $\Delta \simeq S^2 \times \mathbb{R}$, corresponding to the topological structure of the family of Kerr-Newman vacuum solutions one finds for generic black holes that result from gravitational collapse, and which is unique in four dimensions. Different topologies can be found for vacuum spacetimes with more than four dimensions (Empanan & Reall, 2002) explored in (dim. ≥ 5)-gravity. This structure is equipped with an equivalence class $[\mathbf{X}]_{\sim} = \{\mathbf{X} \in T^*\mathcal{M} | \mathbf{X} = c\mathbf{X}', c \in \mathbb{R}^+\}$ of normal, future-pointing ($X_0 > 0$), null ($g(\mathbf{X}, \mathbf{X}) = 0$) forms, with flows that preserve the S^2 -foliation of Δ . The latter statement means that when an object, given on a specific S^2 -section, is transported along the flow of \mathbf{X} , the transition is intrinsic with respect to the leaf. All null normals $\mathbf{X} \in [\mathbf{X}]_{\sim}$ have vanishing expansion θ at Δ

$$\theta_{(\mathbf{X})} = q_{\mu\nu} \nabla^\mu X^\nu \overset{\text{at } \Delta}{\underset{\sim}{=}} 0, \quad (1)$$

where $q_{\mu\nu} = \underset{\sim}{g_{\mu\nu}}$ is the pullback of the metric tensor components $g_{\mu\nu}$ to Δ (the pullback is indicated by the arrow underneath), and, thus, the areas of the 2-sphere sections are constant along the fields \mathbf{X} . The evolution equation for the expansion θ of a general congruence \mathbf{Z} (time-like or null), explaining how the volume of a small ball of matter changes with respect to the time measured by a central, comoving observer, is given by the Raychaudhuri equation

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 + 2\omega^2 - E[\mathbf{Z}]^\mu{}_\mu + \nabla_\mu \dot{Z}^\mu, \quad (2)$$

where $\dot{\theta}$ is the rate of change of the expansion, σ is the shear, which measures the tendency of the initially spherical shape of the ball to become distorted to an ellipsoidal shape, ω is the vorticity, measuring the tendency of the congruences to twist around each other, giving a rotation to the ball, and $E[\mathbf{Z}]^\mu{}_\mu = R_{\mu\nu}Z^\mu Z^\nu$ denotes the tidal tensor, which describes the tidal stresses intrinsic to the ball due to gravitational or other interactions of the ball's constituents. Based on condition (1), it directly follows that $\dot{\theta} = 0$ and, consequently, the S^2 -sections cannot evolve in time, i.e., they are marginally trapped.

At Δ , all general relativistic field equations must hold. Furthermore, the stress-energy tensor $T_{\mu\nu}$ is required to fulfill the dominant energy condition, that is to say that mass-energy fluxes can never be observed to propagate faster than light. In detail, the null congruences of the equivalence class $[\mathbf{X}]_\sim$, which can be interpreted as defining the world lines of a family of ideal observers, have to be future-pointing, the momentum $p^\mu = -T^\mu{}_\nu X^\nu$ has to be future-pointing and causal, and the scalar mass-energy density $\varrho = T_{\mu\nu}X^\mu X^\nu$ has to be non-negative. When the dominant energy condition holds, the Raychaudhuri equation (2) for the null congruence \mathbf{X} becomes

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 - E[\mathbf{X}]^\mu{}_\mu. \quad (3)$$

With condition (1), Eq. (3) yields

$$\sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}X^\mu X^\nu = 0. \quad (4)$$

The inequality $\varrho \geq 0$ implies that $R_{\mu\nu}X^\mu X^\nu \geq 0$, and, therefore, \mathbf{X} is shear-free. As a result of the latter and $\sigma_{\mu\nu} = \underline{\nabla}_{(\mu}X_{\nu)} - q_{\mu\nu}\theta(\mathbf{X})/2$, one infers that $\underline{\nabla}_{(\mu}X_{\nu)} = 0$ at Δ , which is equivalent to $\mathfrak{L}_{\mathbf{X}}q_{\mu\nu} = 0$, where \mathfrak{L} designates the Lie derivative. The last equation can be seen as displaying the time-independence of the intrinsic metric. In order to have a notion of change in space at Δ , one can introduce a unique derivative operator \mathfrak{D} , defined by the covariant derivative ∇ via

$$V^\mu \mathfrak{D}_\mu W^\nu \stackrel{\text{at } \Delta}{=} V^\mu \nabla_\mu W^\nu, \quad (5)$$

with the vector fields \mathbf{V} and \mathbf{W} tangent to Δ , or simply by the pullback of the covariant derivative $\mathfrak{D} = \underline{\nabla}$ to the horizon. The entire isolated horizon geometry is then captured by the pair (q, \mathfrak{D}) , and since the intrinsic metric q is time-independent, it is reasonable to impose the same condition on the derivative operator \mathfrak{D}

$$[\mathfrak{L}_{\mathbf{X}}, \mathfrak{D}] = 0 \quad \forall \quad \mathbf{X} \in [\mathbf{X}]_\sim. \quad (6)$$

Within the meaning of Eqs. (1) and (6), an isolated horizon is in a state of quasi-local equilibrium in terms of the marginally trapped and spatially compact 2-spheres. In the family of Kerr-Newman spacetimes, any Killing horizon that is topologically $S^2 \times \mathbb{R}$, is an isolated horizon. This implies that in particular the outer event horizons of all stationary, axisymmetric black holes are isolated horizons (Ashtekar & Krishnan, 2004). Note that these kinds of spacetimes admit many more isolated horizons, and that the outer event horizon boundaries merely depict the final emergent isolated horizon structure, which is associated to a total equilibrium state without the chance of ever having a non-vanishing matter net flux toward the black hole. The transition of a general isolated horizon state into another takes place when the local equilibrium is disturbed by an infalling matter distribution, forcing the black hole to change and settle down into a new local equilibrium state after a certain amount of time has passed. The former isolated horizon can, thus, no longer be an event horizon, nevertheless both are elements of the family of isolated horizon geometries.

III. CONSTRUCTION OF KERR ISOLATED HORIZON GEOMETRIES

An instructive way toward the $SL(2, \mathbb{C})$ and the $SU(2)$ connection variable formulations of the class of Kerr isolated horizons, which are essential for a description of rotating black holes in loop quantum gravity, is presented. The methodology and logic of how to construct these geometrical structures is given in a descriptive and intuitive manner, using the information about isolated horizon geometries encoded intrinsically in a generic Kerr spacetime.

Step 1: One begins with a stationary, axisymmetric, general relativistic black hole solution of the 2-parameter Kerr family in the standard Boyer-Lindquist representation, and reformulates this geometry in terms of a Newman-Penrose form bundle, i.e., via a null tetrad frame $\{\mathbf{n}, \mathbf{l}, \mathbf{m}, \mathbf{m}^*\}$, adapted to null geodesics of the underlying Kerr spacetime. From a geometrical point of view, in tetrad gravity one now deals with geometrical structures on internal, flat form bundles over the original spacetime manifold. Rewriting general relativity in terms of local tetrads leads to simplifications in many computations of physical quantities.

Step 2: Since the Boyer-Lindquist representation of Kerr spacetime is pathological at the outer event horizon at $r = r_+$, one needs a change to a well-defined coordinate system that properly covers this region. Therefore, a transformation into well-behaved Kruskal-like coordinates is conducted. Additionally, the real-valued

Newman-Penrose basis 1-forms \mathbf{n} and \mathbf{l} are rescaled, using a local Lorentz transformation, in order to obtain a well-defined Newman-Penrose tetrad at the outer event horizon.

Step 3: The proper Newman-Penrose frame is mapped onto an orthonormal, spatiotemporal basis.

Step 4: The first Maurer-Cartan structure equation is solved in order to determine the spin connection 1-form ω . Afterward, the pullback of the spin connection to the family of marginally trapped S^2 -sections (containing all relevant isolated horizon information), which essentially are the intersections of the type II isolated horizon Δ with specific Cauchy surfaces C , $S^2 = \Delta \cap C$, equipped with a form $\mathbf{N} = (\mathbf{n} + \mathbf{l})/\sqrt{2}$, that is aligned to be normal to the 2-spheres in **Step 6**, is evaluated.

Step 5: A local Lorentz transformation is performed, such that, on the one hand, the complex pair of Newman-Penrose 1-forms \mathbf{m}' and \mathbf{m}'^* , arising from $\mathbf{m} \mapsto \mathbf{m}'$ and $\mathbf{m}^* \mapsto \mathbf{m}'^*$, respectively, are tangent to the 2-spheres in order to define the intrinsic derivative operator \mathfrak{D} , and, on the other hand, the real Newman-Penrose 1-form \mathbf{l}' , resulting from $\mathbf{l} \mapsto \mathbf{l}'$, constitutes the equivalence class of expansion-free null normals $\mathbf{l}' \in [\mathbf{l}']_{\sim} = [\mathbf{X}]_{\sim}$ on the isolated horizon. Then, Eq. (6) is satisfied, guaranteeing time-independence on a compact space (quasi-locality). Since the local isolated horizon surface gravity is not completely determined in the isolated horizon framework (because exterior information from the bulk is required), it is adapted, by means of a further local Lorentz transformation, to the standard event horizon surface gravity κ of stationary, axisymmetric black holes, which is defined by the equation

$$\chi^\mu \nabla_\mu \chi_\nu = \kappa \chi_\nu, \quad (7)$$

evaluated at the horizon. The Killing 1-form χ is normal with respect to the horizon, and because no unique notion of this generator exists for isolated horizons, one can choose it in a way such that it coincides with the new null normal \mathbf{l}'' , emanating from $\mathbf{l}' \mapsto \mathbf{l}''$. Moreover, the spin connection is evaluated in the adequately adapted spatiotemporal coordinate system. It is shown in Appendix A that in this frame of reference the isolated horizon boundary conditions (1) and (6), i.e., each element of $[\mathbf{l}'']_{\sim}$ has vanishing expansion and $[\mathfrak{L}_{\mathbf{l}''}, \mathfrak{D}] = 0$ at the horizon Δ , are fulfilled.

Step 6: The spin connection is fixed in the time gauge, which is customary for the construction of the $\text{SL}(2, \mathbb{C})$ and the $\text{SU}(2)$ connection variable formulations and compatible with the normal alignment $\mathbf{N} \mapsto \mathbf{N}_\perp$ with respect to the 2-spheres. Only now, one has a frame with the necessary auxiliary structures that captures the proper degrees of freedom of the type II Kerr isolated horizon.

Step 7: Both the complex, self-dual Ashtekar and the real Ashtekar-Barbero connections \mathbf{A}_+ and \mathbf{A}_γ , given by certain linear combinations of the components of the spin connection ω , and the corresponding curvatures \mathbf{F}_+ and \mathbf{F}_γ are computed at the S^2 -sections of the isolated horizon boundary.

These steps are executed explicitly in the following section.

IV. $\text{SL}(2, \mathbb{C})$ AND $\text{SU}(2)$ CONNECTION VARIABLE FORMULATIONS OF KERR ISOLATED HORIZON GEOMETRIES

A. Newman-Penrose Tetrad Formulation of Kerr Geometry on a Kruskal-Like Coordinate Space

Step 1

A specific class of spacetimes possessing type II isolated horizons is given by the Kerr solutions of the vacuum Einstein equations which include Kerr horizon boundary data. Hence, it is convenient to study the principal equations and properties of type II isolated horizons in the direct context of Kerr spacetimes. The geometry required for the description of a Kerr spacetime, or rather a rotating (Kerr) black hole, is set up by a connected, orientable Lorentz 4-manifold $(\mathcal{M}, g_{\mu\nu})$, which is topologically $S^2 \times \mathbb{R}^2$. The stationary, axisymmetric metric $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ of Kerr geometry in Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$, with the signature convention $(+, -, -, -)$, yields (Kerr, 1963; Boyer & Lindquist, 1967)

$$\mathbf{g} = \frac{\Xi}{\rho^2} (dt - a \sin^2(\theta) d\varphi)^2 - \frac{\sin^2(\theta)}{\rho^2} (\rho_0^2 d\varphi - a dt)^2 - \frac{\rho^2}{\Xi} dr^2 - \rho^2 d\theta^2, \quad (8)$$

where $\Xi = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, with $r > r_+$, $\rho^2 = r^2 + a^2 \cos^2(\theta)$, and $\rho_0^2 = r^2 + a^2$. M denotes the mass, $a = J/M$ is the angular momentum per unit mass, and $r_\pm = M \pm \sqrt{M^2 - a^2}$ are the outer and inner event horizons of the Kerr black hole, respectively. Instead of imposing an orientation and a curvilinear metric structure,

formulated on the cotangent bundle $T^*\mathcal{M}$, one can equivalently use a local frame bundle on \mathcal{M} with structure group $\text{SO}(1, 3)$, thus, considering a description of the geometry in terms of orthonormal Lorentzian tetrad frames at each point of the manifold. It is advantageous to use this local fiber bundle framework in place of the usual metric representation to account for Kerr geometry due to computational simplicity of several physical quantities. The tetrad $\{e_{(a)}\}$ and its dual $\{e^{(a)}\}$ have to fulfill the orthonormality condition $e^{(a)}(e_{(b)}) = \delta_{(b)}^{(a)}$, as well as the metric condition $g(e_{(a)}, e_{(b)}) = \eta_{(a)(b)}$, where $\eta_{(a)(b)}$ is a constant, symmetric matrix. Small Latin indices written in parentheses $((a), (b), \dots)$ are local $\text{SO}(1, 3)$ frame bundle indices, while small Greek indices (μ, ν, \dots) are coordinate space indices. Both sets of indices are ranging over the index values $\{0, 1, 2, 3\}$. Kerr geometry can be appropriately formulated in terms of a Newman-Penrose null tetrad frame consisting of two real-valued basis 1-forms $e_{\text{NP}}^{(0)} = \mathbf{n}$ and $e_{\text{NP}}^{(1)} = \mathbf{l}$, as well as the complex basis 1-form pair $e_{\text{NP}}^{(2)} = \mathbf{m}$ and $e_{\text{NP}}^{(3)} = \mathbf{m}^*$ (Newman & Penrose, 1962, 1966). In this particular representation, the general spacetime metric reads

$$\mathbf{g} = \eta_{(a)(b)} e_{\text{NP}}^{(a)} e_{\text{NP}}^{(b)} = 2(\mathbf{n} \mathbf{l} - \mathbf{m} \mathbf{m}^*), \quad (9)$$

with the internal metric given by

$$\eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (10)$$

The components $e_{\mu \text{NP}}^{(a)}$ of the dual basis 1-forms of the Newman-Penrose tetrad frame describing Kerr geometry, adapted to the basic class of null geodesics defined by the tangent vectors (Chandrasekhar, 1983)

$$\frac{dt}{d\tau} = \frac{\rho_0^2 E}{\Xi}, \quad \frac{dr}{d\tau} = \pm E, \quad \frac{d\theta}{d\tau} = 0, \quad \text{and} \quad \frac{d\varphi}{d\tau} = \frac{a E}{\Xi}, \quad (11)$$

where τ is an affine parameter along the null geodesics and E is a constant, yield

$$\begin{aligned} n_\mu &= \frac{\Xi}{2\rho^2} \left[1, \frac{\rho^2}{\Xi}, 0, -a \sin^2(\theta) \right] \\ l_\mu &= \left[1, -\frac{\rho^2}{\Xi}, 0, -a \sin^2(\theta) \right] \\ m_\mu &= \frac{1}{\sqrt{2}\bar{\rho}^*} \left[ia \sin(\theta), 0, \rho^2, -i\rho_0^2 \sin(\theta) \right] \\ m_\mu^* &= \frac{1}{\sqrt{2}\bar{\rho}} \left[-ia \sin(\theta), 0, \rho^2, i\rho_0^2 \sin(\theta) \right]. \end{aligned} \quad (12)$$

These, together with the contravariant expressions

$$\begin{aligned} n^\mu &= \frac{1}{2\rho^2} \left[\rho_0^2, -\Xi, 0, a \right] \\ l^\mu &= \frac{1}{\Xi} \left[\rho_0^2, \Xi, 0, a \right] \\ m^\mu &= \frac{1}{\sqrt{2}\bar{\rho}^*} \left[ia \sin(\theta), 0, -1, \frac{i}{\sin(\theta)} \right] \\ m^{*\mu} &= -\frac{1}{\sqrt{2}\bar{\rho}} \left[ia \sin(\theta), 0, 1, \frac{i}{\sin(\theta)} \right], \end{aligned} \quad (13)$$

satisfy the null requirements $n_\mu n^\mu = l_\mu l^\mu = m_\mu m^\mu = m_\mu^* m^{*\mu} = 0$, the normalization conditions $n_\mu l^\mu = 1$ and $m_\mu m^{*\mu} = -1$, as well as the orthogonality relations $n_\mu m^\mu = n_\mu m^{*\mu} = l_\mu m^\mu = l_\mu m^{*\mu} = 0$. The radial function ρ^2 is related to the complex quantity $\bar{\rho} = r + ia \cos(\theta)$ via the absolute square $\rho^2 = \bar{\rho} \bar{\rho}^*$.

Step 2

Since Kerr geometry in the Boyer-Lindquist representation is ill-defined at the outer event horizon at $r = r_+$, one needs to find a coordinate system that is well-defined in this region. This can be accomplished using a Kruskal-like, analytic extension of the Boyer-Lindquist coordinates beyond the outer event horizon, which is similar to the well-known complete, analytic extension of Schwarzschild black hole spacetimes. Therefore, firstly, one transforms the Boyer-Lindquist coordinate system into a tortoise null frame $\{u, v, \theta, \tilde{\varphi}_+\}$ by means of

$$u = t - \tilde{r}, \quad v = t + \tilde{r}, \quad \text{and} \quad \tilde{\varphi}_+ = \varphi - \frac{at}{r_+^2 + a^2}, \quad (14)$$

where

$$\tilde{r} = r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln(r - r_+) - \frac{r_-^2 + a^2}{r_+ - r_-} \ln(r - r_-) \quad (15)$$

is the Regge-Wheeler coordinate. Subsequently, in the same manner as in the static, non-rotating case, one introduces ingoing and outgoing Eddington-Finkelstein coordinates

$$U = \exp(-\alpha_+ u) \quad \text{and} \quad V = \exp(\alpha_+ v), \quad (16)$$

with $\alpha_+ = (r_+ - r_-)/(2\rho_{0+}^2)$ and $\rho_{0+}^2 = \rho_0^2(r = r_+)$. In the final step toward the well-behaved Kruskal-like coordinates $\{K, L, \theta, \tilde{\varphi}_+\}$ for Kerr geometry, one defines the variables K and L by linear combinations of U and V according to

$$K = \frac{V - U}{2} = \exp(\alpha_+ \tilde{r}) \sinh(\alpha_+ t) \quad \text{and} \quad L = \frac{V + U}{2} = \exp(\alpha_+ \tilde{r}) \cosh(\alpha_+ t). \quad (17)$$

The Boyer-Lindquist radial coordinate r is related to the Kruskal-like variables K and L via the implicit equation

$$L^2 - K^2 = \exp(2\alpha_+ \tilde{r}) = \Xi\beta(r), \quad (18)$$

where $\beta(r)$ is a real-valued function given by

$$\beta(r) = \frac{\exp(2\alpha_+ r)}{(r - r_-)^{1+(r_+^2+a^2)/\rho_{0+}^2}}. \quad (19)$$

The evaluation of relation (18) on the outer event horizon at $r = r_+$ yields the two solutions $K = \pm L$. Note that the Kruskal-like extension does not cover the region at the inner event horizon at $r = r_-$. Since this particular region is not accessible for exterior observers, it is of no interest for this study, and, thus, poses no problem. The three Boyer-Lindquist coordinate 1-forms $\{dt, dr, d\varphi\}$, in terms of the new Kruskal-like basis, read

$$dt = \frac{LdK - KdL}{\alpha_+(L^2 - K^2)}, \quad dr = \frac{LdL - KdK}{\alpha_+\beta\rho_0^2}, \quad \text{and} \quad d\varphi = d\tilde{\varphi}_+ + \frac{a(LdK - KdL)}{\alpha_+\rho_{0+}^2(L^2 - K^2)}. \quad (20)$$

The components of the Newman-Penrose 1-forms (12) on the Kruskal-like coordinate space, with the basis 1-forms $\{dK, dL, d\theta, d\tilde{\varphi}_+\}$, become

$$\begin{aligned} n_\mu &= \frac{L - K}{2\alpha_+\beta\rho^2} \left[\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) K(L + K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r + r_+}{r - r_-}, \frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) L(L + K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r + r_+}{r - r_-}, 0, -a \sin^2(\theta) \alpha_+(L + K) \right] \\ l_\mu &= \frac{1}{\alpha_+(L - K)} \left[\frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) K(L - K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r + r_+}{r - r_-}, -\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) L(L - K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r + r_+}{r - r_-}, 0, -a \sin^2(\theta) \alpha_+(L - K) \right] \\ m_\mu &= \frac{1}{\sqrt{2}\bar{\rho}^*} \left[-\frac{ia \sin(\theta) L}{\alpha_+\beta\rho_{0+}^2} \times \frac{r + r_+}{r - r_-}, \frac{ia \sin(\theta) K}{\alpha_+\beta\rho_{0+}^2} \times \frac{r + r_+}{r - r_-}, \rho^2, -i\rho_0^2 \sin(\theta) \right] \\ m_\mu^* &= \frac{1}{\sqrt{2}\bar{\rho}} \left[\frac{ia \sin(\theta) L}{\alpha_+\beta\rho_{0+}^2} \times \frac{r + r_+}{r - r_-}, -\frac{ia \sin(\theta) K}{\alpha_+\beta\rho_{0+}^2} \times \frac{r + r_+}{r - r_-}, \rho^2, i\rho_0^2 \sin(\theta) \right]. \end{aligned} \quad (21)$$

In addition to the underlying coordinate system being well-defined, the real-valued Newman-Penrose 1-forms \mathbf{n} and \mathbf{l} have to be rescaled, using a local Lorentz transformation of the form $\mathbf{n} \mapsto \sqrt{2\beta} \rho \mathbf{n} / (L-K)$ and $\mathbf{l} \mapsto (L-K) \mathbf{l} / (\sqrt{2\beta} \rho)$, yielding

$$\begin{aligned} n_\mu &= \frac{\Omega}{\sqrt{2}} \left[\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) K (L+K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, \frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) L (L+K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, 0, -a \sin^2(\theta) \alpha_+ (L+K) \right] \\ l_\mu &= \frac{\Omega}{\sqrt{2}} \left[\frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) K (L-K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, -\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) L (L-K)}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, 0, -a \sin^2(\theta) \alpha_+ (L-K) \right], \end{aligned} \quad (22)$$

with the abbreviation $\Omega = 1/(\alpha_+ \sqrt{\beta} \rho)$, in order to have a regular Newman-Penrose tetrad without pathologies at the outer event horizon for both solutions $K = \pm L$. Since the rescaling is carried out by means of a local Lorentz transformation, it preserves all former Newman-Penrose tetrad frame and Kerr spacetime characteristics.

Step 3

To properly account for Kerr isolated horizons, one is interested in a spatiotemporal tetrad frame for the local description of the horizon, while the Newman-Penrose frame fields, as auxiliary structures, are adapted to satisfy the boundary conditions (1) and (6). Hence, one begins with the determination of a spatiotemporal tetrad, and computes, for the purpose of constructing the $\text{SL}(2, \mathbb{C})$ and the $\text{SU}(2)$ connection variable representations, the $\mathfrak{so}(1, 3)$ -valued spin connection 1-form $\boldsymbol{\omega}$, where $\mathfrak{so}(1, 3)$ denotes the Lie algebra of the structure group $\text{SO}(1, 3)$, by solving the first Maurer-Cartan structure equation $d\mathbf{e} + \boldsymbol{\omega} \wedge \mathbf{e} = 0$. The isolated horizon boundary conditions and the time gauge fixing condition are imposed afterward on the Newman-Penrose frame and, accordingly, on the spatiotemporal tetrad and the spin connection.

The mapping of a generic Newman-Penrose tetrad $\{\mathbf{n}, \mathbf{l}, \mathbf{m}, \mathbf{m}^*\}$ to a spatiotemporal frame of reference $\{e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)}\}$ can be accomplished with the unitary transformation

$$U^{(a)}_{(b)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix}, \quad (23)$$

leading to

$$e^{(0)}_\mu = \frac{n_\mu + l_\mu}{\sqrt{2}}, \quad e^{(1)}_\mu = \frac{n_\mu - l_\mu}{\sqrt{2}}, \quad e^{(2)}_\mu = \frac{m_\mu + m_\mu^*}{\sqrt{2}} \quad \text{and} \quad e^{(3)}_\mu = \frac{m_\mu^* - m_\mu}{\sqrt{2}i}. \quad (24)$$

With the conjugate pair m_μ and m_μ^* from Eq. (21) and the components of the real forms n_μ and l_μ from Eq. (22), Eq. (24) explicitly reads

$$\begin{aligned} e^{(0)}_\mu &= \Omega \left[\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) K^2}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, \frac{a^2 \sin^2(\theta) L K}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, 0, -a \sin^2(\theta) \alpha_+ L \right] \\ e^{(1)}_\mu &= \Omega \left[-\frac{a^2 \sin^2(\theta) L K}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, \frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) L^2}{\rho_{0+}^2 \rho_0^2 \beta} \times \frac{r+r_+}{r-r_-}, 0, -a \sin^2(\theta) \alpha_+ K \right] \\ e^{(2)}_\mu &= \left[\frac{a^2 \sin(2\theta) L}{2\alpha_+ \beta \rho_{0+}^2 \rho^2} \times \frac{r+r_+}{r-r_-}, -\frac{a^2 \sin(2\theta) K}{2\alpha_+ \beta \rho_{0+}^2 \rho^2} \times \frac{r+r_+}{r-r_-}, r, \frac{a \sin(2\theta) \rho_0^2}{2\rho^2} \right] \\ e^{(3)}_\mu &= \left[\frac{a \sin(\theta) r L}{\alpha_+ \beta \rho_{0+}^2 \rho^2} \times \frac{r+r_+}{r-r_-}, -\frac{a \sin(\theta) r K}{\alpha_+ \beta \rho_{0+}^2 \rho^2} \times \frac{r+r_+}{r-r_-}, -a \cos(\theta), \frac{r \sin(\theta) \rho_0^2}{\rho^2} \right]. \end{aligned} \quad (25)$$

In this local Lorentz frame, the internal metric is given by the Minkowski metric with signature (1, 3)

$$\eta_{(a)(b)} = \text{diag}(1, -1, -1, -1). \quad (26)$$

In the following, the computation of the spin connection ω is performed. Note that the specified spin connection is directly evaluated on the outer event horizon at $r = r_+$, or rather $K = L$ in the Kruskal-like coordinates (choosing the positive solution of the boundary equation $K^2 - L^2 = 0$), i.e., one determines the full solution of the first Maurer-Cartan structure equation, where all geometrical degrees of freedom of the 4-manifold \mathcal{M} are still considered, and merely fixes the radial coordinate subsequently. Although one does not have the proper isolated horizon frame at hand yet, quantities evaluated on the outer event horizon boundary are denoted by the index $|\Delta$, because, as stated before, this horizon constitutes the final state of the isolated horizon family. Restricting the radial coordinate to the outer event horizon, the spatiotemporal tetrad (25) yields

$$\begin{aligned}
e^{(0)}_{\mu|\Delta} &= \Omega_+ \left[\frac{\rho_+^2}{\rho_{0+}^2} - \frac{a^2 \sin^2(\theta) r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^6}, \frac{a^2 \sin^2(\theta) r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^6}, 0, -a \sin^2(\theta) \alpha_+ L \right] \\
e^{(1)}_{\mu|\Delta} &= \Omega_+ \left[-\frac{a^2 \sin^2(\theta) r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^6}, \frac{\rho_+^2}{\rho_{0+}^2} + \frac{a^2 \sin^2(\theta) r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^6}, 0, -a \sin^2(\theta) \alpha_+ L \right] \\
e^{(2)}_{\mu|\Delta} &= \left[\frac{a^2 \sin(2\theta) r_+ L \Omega_+^2}{2\rho_{0+}^4}, -\frac{a^2 \sin(2\theta) r_+ L \Omega_+^2}{2\rho_{0+}^4}, r_+, \frac{a \sin(2\theta) \rho_{0+}^2}{2\rho_+^2} \right] \\
e^{(3)}_{\mu|\Delta} &= \left[\frac{a \sin(\theta) r_+^2 L \Omega_+^2}{\rho_{0+}^4}, -\frac{a \sin(\theta) r_+^2 L \Omega_+^2}{\rho_{0+}^4}, -a \cos(\theta), \frac{r_+ \sin(\theta) \rho_{0+}^2}{\rho_+^2} \right].
\end{aligned} \tag{27}$$

The associated exterior derivatives become

$$\begin{aligned}
de^{(0)}_{|\Delta} &= \Omega_+ \left[\frac{L(H_+ - a^2 \sin^2(\theta) r_+)}{\alpha_+ \beta_+ \rho_{0+}^6} dL \wedge dK + \frac{a^2 \sin(2\theta)}{2\rho_{0+}^2} \left[dK - \frac{2r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^4} \left(1 + \frac{a^2 \sin^2(\theta)}{2\rho_+^2} \right) (dL - dK) \right] \wedge d\theta \right. \\
&\quad \left. - a \sin^2(\theta) \alpha_+ \left[dL + \frac{L^2 H_+ \alpha_+ \Omega_+^2}{\rho_{0+}^4} (dL - dK) \right] \wedge d\tilde{\varphi}_+ - a \sin(2\theta) \alpha_+ L \left(1 + \frac{a^2 \sin^2(\theta)}{2\rho_+^2} \right) d\theta \wedge d\tilde{\varphi}_+ \right] \\
de^{(1)}_{|\Delta} &= \Omega_+ \left[\frac{L(H_+ - a^2 \sin^2(\theta) r_+)}{\alpha_+ \beta_+ \rho_{0+}^6} dL \wedge dK + \frac{a^2 \sin(2\theta)}{2\rho_{0+}^2} \left[dL - \frac{2r_+ L^2}{\alpha_+ \beta_+ \rho_{0+}^4} \left(1 + \frac{a^2 \sin^2(\theta)}{2\rho_+^2} \right) (dL - dK) \right] \wedge d\theta \right. \\
&\quad \left. - a \sin^2(\theta) \alpha_+ \left[dK + \frac{L^2 H_+ \alpha_+ \Omega_+^2}{\rho_{0+}^4} (dL - dK) \right] \wedge d\tilde{\varphi}_+ - a \sin(2\theta) \alpha_+ L \left(1 + \frac{a^2 \sin^2(\theta)}{2\rho_+^2} \right) d\theta \wedge d\tilde{\varphi}_+ \right] \\
de^{(2)}_{|\Delta} &= \frac{a^2 \sin(2\theta) r_+ \Omega_+^2}{\rho_{0+}^4} \left[dL \wedge dK - \frac{a \sin^2(\theta) \alpha_+ L \rho_{0+}^2}{\rho_+^2} (dL - dK) \wedge d\tilde{\varphi}_+ \right] \\
&\quad + \frac{L}{\alpha_+ \beta_+ \rho_{0+}^2} \left[1 + \frac{a^2 r_+}{\alpha_+ \rho_{0+}^2 \rho_+^2} \left(1 - \frac{2r_+^2 \sin^2(\theta)}{\rho_+^2} \right) \right] (dL - dK) \wedge d\theta + \frac{a \rho_{0+}^2}{\rho_+^2} \left(1 - \frac{2r_+^2 \sin^2(\theta)}{\rho_+^2} \right) d\theta \wedge d\tilde{\varphi}_+ \\
de^{(3)}_{|\Delta} &= \sin(\theta) \Omega_+^2 \left[\frac{2ar_+^2}{\rho_{0+}^4} dL \wedge dK + \alpha_+ L \left(1 - \frac{2a^2 \sin^2(\theta) r_+^2}{\rho_{0+}^2 \rho_+^2} \right) (dL - dK) \wedge d\tilde{\varphi}_+ \right] \\
&\quad + \frac{r_+ \cos(\theta) \rho_{0+}^2}{\rho_+^2} \left(1 + \frac{2a^2 \sin^2(\theta)}{\rho_+^2} \right) \left[\frac{ar_+ L}{\alpha_+^2 \beta_+ \rho_{0+}^6} (dL - dK) \wedge d\theta + d\theta \wedge d\tilde{\varphi}_+ \right],
\end{aligned} \tag{28}$$

where $H_+ = -r_+ \rho_{0+}^2 + a^2 \rho_+^2 / (\alpha_+ \rho_{0+}^2)$. With Eqs. (27) and (28), one has all necessary quantities to uniquely determine the spin connection.

B. Spin Connection and Isolated Horizon Boundary Conditions

Step 4

A general, unique solution of the first Maurer-Cartan structure equation

$$de^{(a)} + \omega^{(a)}_{(b)} \wedge e^{(b)} = 0 \quad (29)$$

can be obtained in terms of linear combinations of the dual tetrad basis 1-forms $e^{(a)}$ and the structure functions $C_{(a)(b)(c)}$ in the form

$$\omega_{(a)(b)} = \frac{1}{2} [C_{(a)(b)(c)} + C_{(a)(c)(b)} - C_{(b)(c)(a)}] e^{(c)}. \quad (30)$$

The structure functions $C_{(a)(b)(c)}$ are defined by the equation

$$de^{(a)} + \frac{1}{2} C_{(b)(c)}^{(a)} e^{(b)} \wedge e^{(c)} = 0, \quad (31)$$

and they are antisymmetric in their first and second indices. Expressing the Kruskal-like coordinate 1-forms $\{dK, dL, d\theta, d\tilde{\varphi}_+\}$ in the exterior derivatives of the tetrad (28) by means of the 1-forms (27)

$$\begin{aligned} dK &= \frac{\rho_{0+}^2}{\Omega_+ \rho_+^2} e_{|\Delta}^{(0)} + \frac{a \sin(\theta) \alpha_+ L}{\rho_+^2} [a \cos(\theta) e_{|\Delta}^{(2)} + r_+ e_{|\Delta}^{(3)}] \\ dL &= \frac{\rho_{0+}^2}{\Omega_+ \rho_+^2} e_{|\Delta}^{(1)} + \frac{a \sin(\theta) \alpha_+ L}{\rho_+^2} [a \cos(\theta) e_{|\Delta}^{(2)} + r_+ e_{|\Delta}^{(3)}] \\ d\theta &= \frac{1}{\rho_+^2} [r_+ e_{|\Delta}^{(2)} - a \cos(\theta) e_{|\Delta}^{(3)}] \\ d\tilde{\varphi}_+ &= \frac{ar_+ L \Omega_+}{\rho_{0+}^4} [e_{|\Delta}^{(1)} - e_{|\Delta}^{(0)}] + \frac{1}{\rho_{0+}^2 \sin(\theta)} [a \cos(\theta) e_{|\Delta}^{(2)} + r_+ e_{|\Delta}^{(3)}], \end{aligned} \quad (32)$$

one can directly read off the structure functions at $r = r_+$, by using the defining equation (31). One finds that the non-vanishing structure functions are

$$\begin{aligned} C_{(0)(1)}^{(0)} &= C_{(0)(1)}^{(1)} = \frac{LH_+}{\sqrt{\beta_+} \rho_{0+}^2 \rho_+^3}; & C_{(0)(2)}^{(0)} &= C_{(1)(2)}^{(1)} = -\frac{1}{2} C_{(0)(1)}^{(2)} = -\frac{a^2 \sin(2\theta) r_+}{2\rho_+^4} \\ C_{(0)(3)}^{(0)} &= C_{(1)(3)}^{(1)} = \frac{a^3 \sin(2\theta) \cos(\theta)}{2\rho_+^4}; & C_{(1)(2)}^{(0)} &= C_{(0)(2)}^{(1)} = \frac{a^2 \sin(2\theta) \alpha_+}{2\rho_+^2} \\ C_{(1)(3)}^{(0)} &= C_{(0)(3)}^{(1)} = \frac{a \sin(\theta) r_+ \alpha_+}{\rho_+^2}; & C_{(2)(3)}^{(0)} &= C_{(2)(3)}^{(1)} = \frac{2a \cos(\theta) \alpha_+ L \Omega_+}{\rho_+^2} \\ C_{(2)(3)}^{(2)} &= -\frac{a}{\rho_+^2 \sin(\theta)} \left(1 - \frac{2r_+^2 \sin^2(\theta)}{\rho_+^2} \right); & C_{(2)(3)}^{(3)} &= -\frac{r_+ \cot(\theta)}{\rho_+^4} (\rho_{0+}^2 + a^2 \sin^2(\theta)) \\ C_{(0)(2)}^{(2)} &= -C_{(1)(2)}^{(2)} = C_{(0)(3)}^{(3)} & C_{(0)(3)}^{(2)} &= -C_{(1)(3)}^{(2)} = -C_{(0)(2)}^{(3)} \\ &= -C_{(1)(3)}^{(3)} = \frac{r_+ \alpha_+ L \Omega_+}{\rho_+^2}; & &= C_{(1)(2)}^{(3)} = -\frac{a \cos(\theta) \alpha_+ L \Omega_+}{\rho_+^2} \\ C_{(0)(1)}^{(3)} &= \frac{2a \sin(\theta) r_+^2}{\rho_+^4}. \end{aligned} \quad (33)$$

The local information on isolated horizons is encoded in the geometry of the family of the intrinsic, marginally trapped 2-spheres $S^2 = \Delta \cap C$, where C is a Cauchy surface with normal \mathbf{N}_\perp . It is, therefore, sufficient to evaluate the pullback of the spin connection to the S^2 -sections of the Δ -foliation, and afterward to align the Newman-Penrose forms appropriately, according to the isolated horizon boundary conditions.

The pullbacks of the Kerr structure functions (33) and the tetrad (27) yield for the six independent, non-vanishing spin connection components (30)

$$\begin{aligned}
\omega_{\leftarrow}^{(0)(1)} &= \frac{a \sin^2(\theta) \rho_{0+}^2}{\rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) d\tilde{\varphi}_+ \\
\omega_{\leftarrow}^{(0)(2)} &= \omega_{\leftarrow}^{(1)(2)} = \frac{\alpha_+ L \Omega_+}{\rho_+^2} \left([r_+^2 - a^2 \cos^2(\theta)] d\theta + a \sin(2\theta) r_+ d\tilde{\varphi}_+ \right) \\
\omega_{\leftarrow}^{(0)(3)} &= \omega_{\leftarrow}^{(1)(3)} = \frac{\alpha_+ L \Omega_+}{\rho_+^2} \left(-2a \cos(\theta) r_+ d\theta + \sin(\theta) [r_+^2 - a^2 \cos^2(\theta)] d\tilde{\varphi}_+ \right) \\
\omega_{\leftarrow}^{(2)(3)} &= -\frac{a \sin(\theta) r_+}{\rho_+^2} d\theta + \frac{\cos(\theta) \rho_{0+}^4}{\rho_+^4} d\tilde{\varphi}_+,
\end{aligned} \tag{34}$$

where the pullback to S^2 is denoted by the double arrow underneath.

Step 5

The geometric boundary conditions (1) and (6) are imposed on the Newman-Penrose forms given in Eqs. (21) and (22), and, subsequently, on the spatiotemporal tetrad (27) as well as the spin connection (34), by requiring that the complex form $\mathbf{m}'_{|\Delta}$ and its conjugate $\mathbf{m}'_{|\Delta}^*$, emanating from a local Lorentz transformation $\mathbf{m}_{|\Delta} \mapsto \mathbf{m}'_{|\Delta}$ and $\mathbf{m}_{|\Delta}^* \mapsto \mathbf{m}'_{|\Delta}^*$, are tangent to the 2-spheres, and that the real form $\mathbf{l}'_{|\Delta}$, resulting from $\mathbf{l}_{|\Delta} \mapsto \mathbf{l}'_{|\Delta}$, constitutes an element of the equivalence class $[\mathbf{l}']_{\sim}$ of expansion-free null normals to the horizon, fulfilling the condition $[\mathfrak{L}_{\mathbf{l}'}, \mathfrak{D}] = 0$. Moreover, the form $\mathbf{l}''_{|\Delta}$, arising from a further local Lorentz transformation $\mathbf{l}'_{|\Delta} \mapsto \mathbf{l}''_{|\Delta}$, is adjusted to match the generator of the Killing horizon at $r = r_+$. The boundary condition for the complex pair can be explicitly stated by

$$m'_{\mu|\Delta} v^\mu = m'^*_{\mu|\Delta} v^\mu = 0 \quad \forall \quad \mathbf{v} = v^K \partial_K + v^L \partial_L. \tag{35}$$

One can meet this requirement by carrying out a local type I Lorentz transformation (Chandrasekhar, 1983) reading

$$\begin{aligned}
n_{\mu|\Delta} &\mapsto n'_{\mu|\Delta} = n_{\mu|\Delta} + \xi^* m_{\mu|\Delta} + \xi m^*_{\mu|\Delta} + |\xi|^2 l_{\mu|\Delta} \\
l_{\mu|\Delta} &\mapsto l'_{\mu|\Delta} = l_{\mu|\Delta} \\
m_{\mu|\Delta} &\mapsto m'_{\mu|\Delta} = m_{\mu|\Delta} + \xi l_{\mu|\Delta} \\
m^*_{\mu|\Delta} &\mapsto m'^*_{\mu|\Delta} = m^*_{\mu|\Delta} + \xi^* l_{\mu|\Delta},
\end{aligned} \tag{36}$$

with a complex parameter ξ given by

$$\xi = \frac{ia \sin(\theta) r_+ L \Omega_+}{\bar{\rho}_+^* \rho_{0+}^2}, \quad \text{where } \Re(\xi) = -\frac{a^2 \sin(2\theta) r_+ L \Omega_+}{2\rho_{0+}^2 \rho_+^2} \quad \text{and} \quad \Im(\xi) = \frac{a \sin(\theta) r_+^2 L \Omega_+}{\rho_{0+}^2 \rho_+^2}. \tag{37}$$

As a result of this transformation, one obtains the following components of the primed Newman-Penrose 1-forms

$$\begin{aligned}
n'_{\mu|\Delta} &= \frac{\rho_+^2 \Omega_+}{\sqrt{2} \rho_{0+}^2} \left[1 - \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), 1 + \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), 0, -\frac{2a \sin^2(\theta) L \rho_{0+}^2}{\rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right], \\
l'_{\mu|\Delta} &= \frac{\rho_+^2 \Omega_+}{\sqrt{2} \rho_{0+}^2} [1, -1, 0, 0], \quad m'_{\mu|\Delta} = \frac{1}{\sqrt{2} \bar{\rho}_+^*} [0, 0, \rho_+^2, -i \rho_{0+}^2 \sin(\theta)], \quad \text{and} \quad m'^*_{\mu|\Delta} = \frac{1}{\sqrt{2} \bar{\rho}_+} [0, 0, \rho_+^2, i \rho_{0+}^2 \sin(\theta)],
\end{aligned} \tag{38}$$

in which $\mathbf{m}'_{|\Delta}$ and $\mathbf{m}'_{|\Delta}^*$ are tangent to the foliation of 2-spheres. Since (36), together with (37), automatically aligns $\mathbf{l}'_{|\Delta} \perp \Delta$ correctly, one only has to impose information from the bulk structure on $\mathbf{l}'_{|\Delta}$. Thus, the local isolated horizon surface gravity κ_{IH} is adjusted to coincide with the surface gravity κ of stationary Kerr black holes defined

at infinity, by stipulating that the new null normal $l''_{|\Delta}$ is the Killing generator $\chi_{|\Delta}$ associated with the stationarity and the axial symmetry of Kerr spacetime

$$l''_{|\Delta} = l''_{\mu|\Delta} dx^\mu = \chi_{|\Delta} = [g_{\mu\nu} \chi^\nu]_{|\Delta} dx^\mu = \frac{\alpha_+ L \Omega_+^2 \rho_+^4}{\rho_{0+}^4} (dK - dL), \quad (39)$$

where the Killing vector in Boyer-Lindquist coordinates reads $\chi_{|\Delta}^\mu = [1, 0, 0, \Omega_H]$. The quantity Ω_H is the angular velocity of the outer event horizon. The covariant components of the Killing generator $\chi_{|\Delta}$ can be given in the form

$$\chi_{\mu|\Delta} = a n'_{\mu|\Delta} + b l'_{\mu|\Delta} + c m'_{\mu|\Delta} + d m'^*_{\mu|\Delta}, \quad (40)$$

with $a = \chi^\nu l'_{\nu|\Delta} = 0$, $b = \chi^\nu n'_{\nu|\Delta} = \sqrt{2} \alpha_+ L \Omega_+ \rho_+^2 / \rho_{0+}^2$, $c = -\chi^\nu m'_{\nu|\Delta} = 0$, and $d = -\chi^\nu m'^*_{\nu|\Delta} = 0$. Then, it directly follows that

$$l''_{\mu|\Delta} = \frac{\sqrt{2} \alpha_+ L \Omega_+ \rho_+^2}{\rho_{0+}^2} l'_{\mu|\Delta}. \quad (41)$$

A local type III Lorentz transformation

$$\begin{aligned} n'_{\mu|\Delta} &\mapsto n''_{\mu|\Delta} = \varepsilon^{-1} n'_{\mu|\Delta} \\ l'_{\mu|\Delta} &\mapsto l''_{\mu|\Delta} = \varepsilon l'_{\mu|\Delta} \\ m'_{\mu|\Delta} &\mapsto m''_{\mu|\Delta} = e^{i\theta} m'_{\mu|\Delta} \\ m'^*_{\mu|\Delta} &\mapsto m''^*_{\mu|\Delta} = e^{-i\theta} m'^*_{\mu|\Delta} \end{aligned} \quad (42)$$

can be used in order to describe the transition from the primed to the double-primed system as laid down in Eq. (41). The transformation parameter can be read off explicitly as $\varepsilon = \sqrt{2} \alpha_+ L \Omega_+ \rho_+^2 / \rho_{0+}^2$, while the second parameter θ is arbitrary, and, therefore, one takes the most natural choice $\theta = 0$. If one were not to restrict θ to a fixed value in the type III transformation, one would introduce an additional but redundant degree of freedom, that merely results in a multiplicative phase in the connection and the curvature. In the double-primed system, the components of the Newman-Penrose 1-forms, with a real form $l''_{|\Delta}$ that is normal to the horizon and adjusted to generate the outer event horizon of the Kerr black hole, and a complex pair $m''_{|\Delta}$ and $m''^*_{|\Delta}$ that is tangent to the horizon, yield

$$\begin{aligned} n''_{\mu|\Delta} &= \frac{1}{2\alpha_+ L} \left[1 - \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), 1 + \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), 0, -\frac{2a \sin^2(\theta) L \rho_{0+}^2}{\rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right], \\ l''_{\mu|\Delta} &= \frac{\alpha_+ L \Omega_+^2 \rho_+^4}{\rho_{0+}^4} [1, -1, 0, 0], \quad m''_{\mu|\Delta} = \frac{1}{\sqrt{2} \rho_+^*} [0, 0, \rho_+^2, -i \rho_{0+}^2 \sin(\theta)], \quad \text{and} \quad m''^*_{\mu|\Delta} = \frac{1}{\sqrt{2} \rho_+} [0, 0, \rho_+^2, i \rho_{0+}^2 \sin(\theta)]. \end{aligned} \quad (43)$$

One obtains the corresponding contravariant expressions

$$\begin{aligned} n''^{\mu}_{|\Delta} &= \frac{\rho_{0+}^4}{2\alpha_+ L \Omega_+^2 \rho_+^4} \left[1 + \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), -1 + \frac{a^2 \sin^2(\theta) r_+ L^2 \Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+ + \frac{r_+}{\rho_+^2} \right), 0, 0 \right] \\ l''^{\mu}_{|\Delta} &= \alpha_+ L [1, 1, 0, 0] \\ m''^{\mu}_{|\Delta} &= \frac{1}{\sqrt{2} \rho_+^*} \left[i a \sin(\theta) L \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), i a \sin(\theta) L \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), -1, \frac{i \rho_+^2}{\rho_{0+}^2 \sin(\theta)} \right] \\ m''^{*\mu}_{|\Delta} &= -\frac{1}{\sqrt{2} \rho_+} \left[i a \sin(\theta) L \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), i a \sin(\theta) L \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), 1, \frac{i \rho_+^2}{\rho_{0+}^2 \sin(\theta)} \right]. \end{aligned} \quad (44)$$

In a matrix representation, the composition of both Lorentz transformations, (36) and (42), becomes

$$\Lambda^{(a)}_{(b)} = \begin{pmatrix} \varepsilon^{-1} & \varepsilon^{-1}|\xi|^2 & \varepsilon^{-1}\xi^* & \varepsilon^{-1}\xi \\ 0 & \varepsilon & 0 & 0 \\ 0 & \xi & 1 & 0 \\ 0 & \xi^* & 0 & 1 \end{pmatrix}. \quad (45)$$

In the spatiotemporal frame of reference, spanned by the basis tetrad (27), the composite transformation reads

$$\tilde{\Lambda}^{(a)}_{(b)} = \begin{pmatrix} \frac{\varepsilon + \varepsilon^{-1}(1 + |\xi|^2)}{2} & \frac{-\varepsilon + \varepsilon^{-1}(1 - |\xi|^2)}{2} & \varepsilon^{-1}\Re(\xi) & -\varepsilon^{-1}\Im(\xi) \\ \frac{-\varepsilon + \varepsilon^{-1}(1 + |\xi|^2)}{2} & \frac{\varepsilon + \varepsilon^{-1}(1 - |\xi|^2)}{2} & \varepsilon^{-1}\Re(\xi) & -\varepsilon^{-1}\Im(\xi) \\ \Re(\xi) & -\Re(\xi) & 1 & 0 \\ -\Im(\xi) & \Im(\xi) & 0 & 1 \end{pmatrix}. \quad (46)$$

Consequently, under a change of basis, $e \mapsto \tilde{e} = \tilde{\Lambda}e$, with the local $\mathfrak{so}(1,3)$ -valued matrix function $\tilde{\Lambda}$ given by Eq. (46), the new spatiotemporal tetrad \tilde{e} yields

$$\begin{aligned} \tilde{e}^{(0)}_{\mu|\Delta} &= \frac{1}{2\sqrt{2}\alpha_+L} \left[1 + \frac{L^2\Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+^2\rho_+^4 - a^2 \sin^2(\theta)r_+ \left[2\alpha_+ + \frac{r_+}{\rho_+^2} \right] \right), 1 - \frac{L^2\Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+^2\rho_+^4 - a^2 \sin^2(\theta)r_+ \left[2\alpha_+ + \frac{r_+}{\rho_+^2} \right] \right), \right. \\ &\quad \left. , 0, -\frac{2a \sin^2(\theta)L\rho_{0+}^2}{\rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right], \\ \tilde{e}^{(1)}_{\mu|\Delta} &= \frac{1}{2\sqrt{2}\alpha_+L} \left[1 - \frac{L^2\Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+^2\rho_+^4 + a^2 \sin^2(\theta)r_+ \left[2\alpha_+ + \frac{r_+}{\rho_+^2} \right] \right), 1 + \frac{L^2\Omega_+^2}{\rho_{0+}^4} \left(2\alpha_+^2\rho_+^4 + a^2 \sin^2(\theta)r_+ \left[2\alpha_+ + \frac{r_+}{\rho_+^2} \right] \right), \right. \\ &\quad \left. , 0, -\frac{2a \sin^2(\theta)L\rho_{0+}^2}{\rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right], \\ \tilde{e}^{(2)}_{\mu|\Delta} &= \left[0, 0, r_+, \frac{a \sin(2\theta)\rho_{0+}^2}{2\rho_+^2} \right], \quad \text{and} \quad \tilde{e}^{(3)}_{\mu|\Delta} = \left[0, 0, -a \cos(\theta), \frac{r_+ \sin(\theta)\rho_{0+}^2}{\rho_+^2} \right]. \end{aligned} \quad (47)$$

The components of the spin connection pullback (34) are Lorentz-transformed as well, according to the transformation law for connections

$$\omega \mapsto \tilde{\omega} = \tilde{\Lambda} \omega \tilde{\Lambda}^{-1} - (d\tilde{\Lambda})\tilde{\Lambda}^{-1}, \quad (48)$$

leading to

$$\begin{aligned} \tilde{\omega}^{(0)(1)}_{\leftarrow} &= \omega^{(0)(1)}_{\leftarrow} + \frac{a^2 \sin(2\theta)}{2\rho_+^2} d\theta \\ \tilde{\omega}^{(0)(2)}_{\leftarrow} &= \tilde{\omega}^{(1)(2)}_{\leftarrow} = \frac{1}{\varepsilon} \left[\omega^{(0)(2)}_{\leftarrow} - \Re(\xi) \omega^{(0)(1)}_{\leftarrow} + \Im(\xi) \omega^{(2)(3)}_{\leftarrow} + d\Re(\xi) \right] \\ \tilde{\omega}^{(0)(3)}_{\leftarrow} &= \tilde{\omega}^{(1)(3)}_{\leftarrow} = \frac{1}{\varepsilon} \left[\omega^{(0)(3)}_{\leftarrow} + \Im(\xi) \omega^{(0)(1)}_{\leftarrow} + \Re(\xi) \omega^{(2)(3)}_{\leftarrow} - d\Im(\xi) \right] \\ \tilde{\omega}^{(2)(3)}_{\leftarrow} &= \omega^{(2)(3)}_{\leftarrow}. \end{aligned} \quad (49)$$

Step 6

The normal alignment $\mathbf{N} \mapsto \widetilde{\mathbf{N}} \mapsto \mathbf{N}_\perp$ of the 1-form $\widetilde{\mathbf{N}} = \widetilde{\boldsymbol{\varepsilon}}^{(0)} = (\mathbf{n}'' + \mathbf{l}'')/\sqrt{2}$ with regard to the 2-spheres is consistent with the time gauge fixing procedure, where a time-like basis form is adjusted to be normal to a family of space-like 3-surfaces. This is the common gauge for the formulations of both complex, self-dual Ashtekar and real Ashtekar-Barbero connections (see, e.g., the introduction to loop quantum gravity by Doná & Speziale (2010)). Therefore, the frame (47) is changed at the intersections $S^2 = \Delta \cap C$, such that the new time-like form is normal to the S^2 -sections of the Δ -foliation

$$\bar{e}^{(0)}_{\mu|\Delta} w^\mu = 0 \quad \forall \quad \mathbf{w} = w^\theta \partial_\theta + w^{\tilde{\varphi}^+} \partial_{\tilde{\varphi}^+}. \quad (50)$$

One can use a local Lorentz transformation

$$\bar{\Lambda}^{(a)}_{(b)} = \begin{pmatrix} 1 + \frac{|\psi|^2}{2} & -\frac{|\psi|^2}{2} & \Re(\psi) & -\Im(\psi) \\ \frac{|\psi|^2}{2} & 1 - \frac{|\psi|^2}{2} & \Re(\psi) & -\Im(\psi) \\ \Re(\psi) & -\Re(\psi) & 1 & 0 \\ -\Im(\psi) & \Im(\psi) & 0 & 1 \end{pmatrix}, \quad (51)$$

with the parameter

$$\psi = -\frac{ia \sin(\theta)}{\sqrt{2}\alpha_+ \bar{\rho}_+^*} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), \quad \text{where} \quad \Re(\psi) = \frac{a^2 \sin(2\theta)}{2\sqrt{2}\alpha_+ \rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \quad \text{and} \quad \Im(\psi) = -\frac{ar_+ \sin(\theta)}{\sqrt{2}\alpha_+ \rho_+^2} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), \quad (52)$$

in order to define this particular gauge fixing, which leads to the following dual basis 1-form components of the tetrad of interest \bar{e}

$$\begin{aligned} \bar{e}^{(0)}_{\mu|\Delta} &= \frac{1}{2\sqrt{2}\alpha_+ L} \left[1 + \frac{2L^2}{\beta_+ \rho_{0+}^2} \left(1 - \frac{a^2 \sin^2(\theta)}{2\rho_{0+}^2} \right), 1 - \frac{2L^2}{\beta_+ \rho_{0+}^2} \left(1 - \frac{a^2 \sin^2(\theta)}{2\rho_{0+}^2} \right), 0, 0 \right] \\ \bar{e}^{(1)}_{\mu|\Delta} &= \frac{1}{2\sqrt{2}\alpha_+ L} \left[1 - \frac{2L^2}{\beta_+ \rho_{0+}^2} \left(1 - \frac{3a^2 \sin^2(\theta)}{2\rho_{0+}^2} \right), 1 + \frac{2L^2}{\beta_+ \rho_{0+}^2} \left(1 - \frac{3a^2 \sin^2(\theta)}{2\rho_{0+}^2} \right), 0, 0 \right] \\ \bar{e}^{(2)}_{\mu|\Delta} &= \left[\frac{a^2 \sin(2\theta) L \Omega_+^2 \rho_+^2}{2\rho_{0+}^4} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), -\frac{a^2 \sin(2\theta) L \Omega_+^2 \rho_+^2}{2\rho_{0+}^4} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), r_+, \frac{a \sin(2\theta) \rho_{0+}^2}{2\rho_+^2} \right] \\ \bar{e}^{(3)}_{\mu|\Delta} &= \left[\frac{ar_+ \sin(\theta) L \Omega_+^2 \rho_+^2}{\rho_{0+}^4} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), -\frac{ar_+ \sin(\theta) L \Omega_+^2 \rho_+^2}{\rho_{0+}^4} \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right), -a \cos(\theta), \frac{r_+ \sin(\theta) \rho_{0+}^2}{\rho_+^2} \right]. \end{aligned} \quad (53)$$

The spin connection components (49) become

$$\begin{aligned} \bar{\omega}_{\leftarrow}^{(0)(1)} &= \bar{\omega}_{\leftarrow}^{(0)(1)} + \frac{a^2 \sin(2\theta)}{2\rho_+^2} d\theta \\ \bar{\omega}_{\leftarrow}^{(2)(3)} &= \bar{\omega}_{\leftarrow}^{(2)(3)} \\ \bar{\omega}_{\leftarrow}^{(0)(2)} &= \bar{\omega}_{\leftarrow}^{(1)(2)} = - \left[\Re(\psi) + \frac{\Re(\xi)}{\varepsilon} \right] \bar{\omega}_{\leftarrow}^{(0)(1)} + \frac{1}{\varepsilon} \bar{\omega}_{\leftarrow}^{(0)(2)} + \left[\Im(\psi) + \frac{\Im(\xi)}{\varepsilon} \right] \bar{\omega}_{\leftarrow}^{(2)(3)} + \frac{1}{\varepsilon} \left[\Re(\psi) d\varepsilon + d\Re(\xi) \right] + d\Re(\psi) \\ \bar{\omega}_{\leftarrow}^{(0)(3)} &= \bar{\omega}_{\leftarrow}^{(1)(3)} = \left[\Im(\psi) + \frac{\Im(\xi)}{\varepsilon} \right] \bar{\omega}_{\leftarrow}^{(0)(1)} + \frac{1}{\varepsilon} \bar{\omega}_{\leftarrow}^{(0)(3)} + \left[\Re(\psi) + \frac{\Re(\xi)}{\varepsilon} \right] \bar{\omega}_{\leftarrow}^{(2)(3)} - \frac{1}{\varepsilon} \left[\Im(\psi) d\varepsilon + d\Im(\xi) \right] - d\Im(\psi). \end{aligned} \quad (54)$$

The tetrad (53) and the spin connection components (54) capture the degrees of freedom of Kerr isolated horizon geometries, and, thus, are the proper quantities for the description of their classical kinematics. They are vital in the computation of the Ashtekar and the Ashtekar-Barbero boundary connections and the associated curvatures, respectively.

C. Self-Dual $\text{SL}(2, \mathbb{C})$ and $\text{SU}(2)$ Connections and Curvatures

Step 7

Up to this point, general relativity was described in terms of a tetrad and a real-valued $\text{SO}(1, 3)$ connection ω . One can construct a Hamiltonian (ADM) formulation of general relativity (for a detailed review on the topic, see Rovelli (1991)), using a triad and a $\text{SL}(2, \mathbb{C}) \simeq \text{SO}(3, \mathbb{C})$ connection A_+ , and, thus, one then operates on a 3-dimensional internal frame bundle (with indices $((i), (j), \dots)$) instead of the original Lorentzian frame bundle. In more detail, one deals with a triad $e^{(i)}$ and three complex, self-dual 1-forms $A_+^{(i)}$ in place of a tetrad $e^{(a)}$ and six real-valued 1-forms $\omega^{(a)(b)}$ as configuration variables for general relativity. The $\text{SL}(2, \mathbb{C})$ connection 1-forms $A_+^{(i)}$ are defined by means of the self-dual projection homomorphism $P_+^{(i)}{}_{(j)(k)} = \epsilon^{(i)}{}_{(j)(k)}/2$ and $P_+^{(i)}{}_{(0)(k)} = -P_+^{(i)}{}_{(k)(0)} = i\delta_{(k)}^{(i)}/2$ as

$$A_+^{(i)}{}_\mu = P_+^{(i)}{}_{(a)(b)} \omega^{(a)(b)}{}_\mu. \quad (55)$$

Note that, since the $\mathfrak{so}(1, 3; \mathbb{R})$ Lie algebra of the original internal Lorentzian frame bundle cannot be decomposed into self-dual and anti-self-dual algebras, one has to consider the complexification $\mathfrak{so}(1, 3; \mathbb{C})$, whereas in this case, such a decomposition, $\mathfrak{so}(1, 3; \mathbb{C}) = \mathfrak{so}(1, 3; \mathbb{C})_+ \oplus \mathfrak{so}(1, 3; \mathbb{C})_-$, is available. It is important to stress that it does not present a problem to extend the fields e, ω and the metric η , which were initially defined on a real frame bundle, to sections of a complexified frame bundle over the spacetime. The self-dual and anti-self-dual parts of the complexified Lie algebra are given by $\mathfrak{so}(1, 3; \mathbb{C})_+ = \{\tau \in \mathfrak{so}(1, 3; \mathbb{C}) \mid \star \tau = i\tau\}$ and $\mathfrak{so}(1, 3; \mathbb{C})_- = \{\tau \in \mathfrak{so}(1, 3; \mathbb{C}) \mid \star \tau = -i\tau\}$, respectively, where τ is an eigenfunction of the Hodge star operator. The real connection ω can, thus, be split into a self-dual and an anti-self-dual part $\omega = \omega_+ + \omega_-$, where both ω_+ and ω_- contain the same information as ω itself. Therefore, concentrating on the 3-dimensional projection of the self-dual part of the spin connection, one can construct a triad formulation for Hamiltonian general relativity in terms of the $\text{SL}(2, \mathbb{C})$ connection 1-forms (55). It is customary to write the self-dual connection via the quantities $\Gamma^{(i)} = \epsilon^{(i)}{}_{(j)(k)} \omega^{(j)(k)}/2$ and $K^{(i)} = \omega^{(0)(i)}$, with the convention $\epsilon^{(1)(2)(3)} = 1$. Then, the connection components (55) become

$$A_+^{(i)} = \Gamma^{(i)} + iK^{(i)}. \quad (56)$$

The corresponding self-dual curvature, which is simply the 3-dimensional, self-dual projection of the curvature R , reads component-wise

$$F_+^{(i)} = F^{(i)}(A_+) = dA_+^{(i)} + \frac{1}{2} \epsilon^{(i)}{}_{(j)(k)} A_+^{(j)} \wedge A_+^{(k)}. \quad (57)$$

This is a peculiarity of the self-dual formalism and not valid in general. Using the proper spin connection components (54), one can compute the pullback of the Ashtekar connection components to the S^2 -sections of the Kerr isolated horizon foliation

$$\begin{aligned} \bar{A}_+^{(1)} &= -\frac{a \sin(\theta)}{\bar{\rho}_+} d\theta + \frac{\rho_{0+}^2}{\rho_+^2} \left[\cos(\theta) + ia \sin^2(\theta) \left(\alpha_+ + \frac{1}{\bar{\rho}_+} \right) \right] d\tilde{\varphi}_+ \\ \bar{A}_+^{(2)} &= i \bar{A}_+^{(3)} = \frac{1}{\sqrt{2}\bar{\rho}_+} \left[\left(r_+ + \frac{a^2 \sin^2(\theta)}{\bar{\rho}_+} \right) i d\theta - \frac{\sin(\theta) \rho_{0+}^2}{\rho_+^2} \left(r_+ - a^2 \sin^2(\theta) \left[\alpha_+ + \frac{1}{\bar{\rho}_+} \right] \right) d\tilde{\varphi}_+ \right]. \end{aligned} \quad (58)$$

Substituting this into Eq. (57) leads to the curvature pullback

$$\begin{aligned} \bar{F}_+^{(1)} &= 2 \sin(\theta) \rho_{0+}^2 \Psi_{2|\Delta}^* d\theta \wedge d\tilde{\varphi}_+ \\ \bar{F}_+^{(2)} &= i \bar{F}_+^{(3)} = -\frac{3ia \sin^2(\theta) \rho_{0+}^2}{\sqrt{2}\bar{\rho}_+} \Psi_{2|\Delta}^* d\theta \wedge d\tilde{\varphi}_+, \end{aligned} \quad (59)$$

where

$$\Psi_{2|\Delta}^* = -\frac{\rho_{0+}^2}{2r_+ \bar{\rho}_+^3} = \frac{M}{\rho_+^6} \left(-r_+ \left[r_+^2 - 3a^2 \cos^2(\theta) \right] + ia \cos(\theta) \left[3r_+^2 - a^2 \cos^2(\theta) \right] \right) \quad (60)$$

is the complex conjugate of the second Weyl scalar at the horizon. The curvature (59) can be rewritten in terms of the 3-dimensional projection of the Plebanski 2-form $\bar{\Sigma}$, which, in the time gauge, is given by

$$\bar{\Sigma}_{\leftarrow}^{(i)} = \frac{1}{2} \epsilon_{(j)(k)}^{(i)} \bar{e}_{\leftarrow}^{(j)} \wedge \bar{e}_{\leftarrow}^{(k)}. \quad (61)$$

With

$$\bar{\Sigma}_{\leftarrow}^{(1)} = \rho_{0+}^2 \sin(\theta) d\theta \wedge d\tilde{\varphi}_+ \quad \text{and} \quad \bar{\Sigma}_{\leftarrow}^{(2)} = \bar{\Sigma}_{\leftarrow}^{(3)} = 0, \quad (62)$$

the Ashtekar curvature components (59) read

$$\begin{aligned} \bar{F}_{\leftarrow+}^{(1)} &= 2\Psi_{2|\Delta}^* \bar{\Sigma}_{\leftarrow}^{(1)} \\ \bar{F}_{\leftarrow+}^{(2)} &= i \bar{F}_{\leftarrow+}^{(3)} = -\frac{3ia \sin(\theta)}{\sqrt{2}\rho_+} \Psi_{2|\Delta}^* \bar{\Sigma}_{\leftarrow}^{(1)}. \end{aligned} \quad (63)$$

The computation of the pullback of the SU(2) Ashtekar-Barbero connection components $A_\gamma^{(i)} = \Gamma^{(i)} + \gamma K^{(i)}$, with the fundamental parameter of loop quantum gravity, namely the Immirzi parameter $\gamma \in \mathbb{R}^+$, results in

$$\begin{aligned} \bar{A}_{\leftarrow\gamma}^{(1)} &= -\frac{a \sin(\theta)}{\rho_+^2} (r_+ - \gamma a \cos(\theta)) d\theta + \frac{\rho_{0+}^2}{\rho_+^2} \left[\frac{\rho_{0+}^2}{\rho_+^2} \cos(\theta) + \gamma a \sin^2(\theta) \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right] d\tilde{\varphi}_+ \\ \bar{A}_{\leftarrow\gamma}^{(2)} &= \frac{1}{\sqrt{2}\rho_+^4} \left[\left(\rho_{0+}^2 r_+ [\gamma r_+ + a \cos(\theta)] + a^3 \sin^2(\theta) \cos(\theta) [r_+ - \gamma a \cos(\theta)] \right) d\theta \right. \\ &\quad \left. - \frac{\rho_{0+}^2 \sin(\theta)}{\rho_+^2} \left(\rho_+^2 \left[r_+ - a^2 \sin^2(\theta) \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right] [r_+ - \gamma a \cos(\theta)] + a^3 \sin^2(\theta) \cos(\theta) [\gamma r_+ + a \cos(\theta)] \right) d\tilde{\varphi}_+ \right] \\ \bar{A}_{\leftarrow\gamma}^{(3)} &= \frac{1}{\sqrt{2}\rho_+^4} \left[\left(\rho_{0+}^2 r_+ [r_+ - \gamma a \cos(\theta)] - a^3 \sin^2(\theta) \cos(\theta) [\gamma r_+ + a \cos(\theta)] \right) d\theta \right. \\ &\quad \left. + \frac{\rho_{0+}^2 \sin(\theta)}{\rho_+^2} \left(\rho_+^2 \left[r_+ - a^2 \sin^2(\theta) \left(\alpha_+ + \frac{r_+}{\rho_+^2} \right) \right] [\gamma r_+ + a \cos(\theta)] - a^3 \sin^2(\theta) \cos(\theta) [r_+ - \gamma a \cos(\theta)] \right) d\tilde{\varphi}_+ \right]. \end{aligned} \quad (64)$$

The associated curvature $\mathbf{F}_\gamma = \mathbf{F}(\mathbf{A}_\gamma)$ can be expressed by means of the self-dual curvature \mathbf{F}_+ , yielding

$$\bar{F}_{\leftarrow\gamma}^{(i)} = \bar{F}_{\leftarrow+}^{(i)} + \frac{1+\gamma^2}{2} \epsilon_{(j)(k)}^{(i)} \bar{K}_{\leftarrow}^{(j)} \wedge \bar{K}_{\leftarrow}^{(k)} + (\gamma - i) \left(d\bar{K}_{\leftarrow}^{(i)} + \epsilon_{(j)(k)}^{(i)} \bar{\Gamma}_{\leftarrow}^{(j)} \wedge \bar{K}_{\leftarrow}^{(k)} \right). \quad (65)$$

One finds, using Eq. (63), the Ashtekar-Barbero curvature expressions

$$\begin{aligned} \bar{F}_{\leftarrow\gamma}^{(1)} &= 2 \left[\Re(\Psi_{2|\Delta}^*) + \gamma \Im(\Psi_{2|\Delta}^*) + \frac{1+\gamma^2}{4\rho_+^6} \left(r_+^4 - a^4 \cos^4(\theta) + \frac{a^2 \rho_{0+}^2}{2} (3 \cos^2(\theta) - 1) \right) \right] \bar{\Sigma}_{\leftarrow}^{(1)} \\ \bar{F}_{\leftarrow\gamma}^{(2)} &= -\frac{3a \sin(\theta)}{\sqrt{2}\rho_+^2} \left[\left(\gamma r_+ + a \cos(\theta) \right) \Re(\Psi_{2|\Delta}^*) + \left(\gamma a \cos(\theta) - r_+ \right) \Im(\Psi_{2|\Delta}^*) + \frac{a \cos(\theta) (1+\gamma^2)}{6\rho_+^4} (5r_+^2 - a^2 \cos(2\theta)) \right] \bar{\Sigma}_{\leftarrow}^{(1)} \\ \bar{F}_{\leftarrow\gamma}^{(3)} &= -\frac{3a \sin(\theta)}{\sqrt{2}\rho_+^2} \left[\left(r_+ - \gamma a \cos(\theta) \right) \Re(\Psi_{2|\Delta}^*) + \left(\gamma r_+ + a \cos(\theta) \right) \Im(\Psi_{2|\Delta}^*) + \frac{r_+ (1+\gamma^2)}{6\rho_+^4} (3r_+^2 - a^2 [1 + 2 \cos(2\theta)]) \right] \bar{\Sigma}_{\leftarrow}^{(1)}, \end{aligned} \quad (66)$$

where $\Psi_{2|\Delta}^*$ is defined by Eq. (60) and the pullback of the Plebanski 2-form component $\bar{\Sigma}_{\leftarrow}^{(1)}$ by Eq. (62).

V. TOWARD QUANTUM KERR ISOLATED HORIZONS

In loop quantum gravity, one considers a quantum isolated horizon to be a topological system, which is described by the degrees of freedom of its boundary defects (Rovelli, 1996; Ghosh & Perez, 2011). A more descriptive view on these structures is given in terms of an ensemble (gas) of distinguishable, non-interacting particles on a boundary surface. The topological boundary defects/particles have their origin in the polymer-like bulk spin network excitations, representing the quantum-gravitational states of the black hole, that puncture the horizon surface (see FIG. 1).

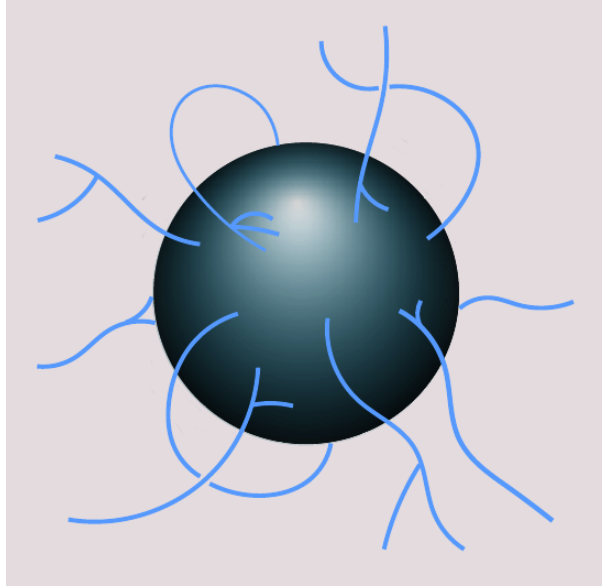


FIG. 1: Microscopic realization of a quantum isolated horizon geometry as boundary surface with topological defects (punctures), which emerges from statistical mechanical considerations in loop quantum gravity. Quantum-gravitational, polymer-like excitations (blue lines) in the spacetime bulk puncture the horizon (sphere) and account for a discrete set of area values.

They are related to particular spin quantum numbers of the spin networks. The quantum geometry of the horizon boundary, or rather the energy spectrum of the gas, is measured by the discrete kinematical loop quantum gravity area operator \hat{A} , with the eigenvalue spectrum (De Pietri & Rovelli, 1996)

$$\hat{A}|j_p, m_p\rangle = 8\pi l_p^2 \gamma \sum_p \sqrt{j_p(j_p + 1)} |j_p, m_p\rangle, \quad (67)$$

where j_p and m_p denote the spin and the magnetic quantum numbers associated to the p th puncture, respectively, taking on values in the half-integer set $\mathbb{N}/2$ and in the set $\{-j_p, -j_p + 1, \dots, j_p - 1, j_p\}$. The eigenstates $|j_p, m_p\rangle$ are the aforementioned quantum-gravitational spin network states. The construction of a quantum theory of Kerr isolated horizons is possible and meaningful in this context. Therefore, a quantum model of rotating Kerr isolated horizons can be based on a $SU(2)$ boundary Chern-Simons theory for the description of the degrees of freedom in the classical phase space (Frodden et al., 2012). As a key input, one requires a boundary constraint equation that relates the pullback of the real Ashtekar-Barbero curvature \mathbf{F}_γ to the pullback of the projection of the Plebanski field Σ in the specific form

$$\mathbf{F}_\gamma^{(i)} = \text{const.} \times \Sigma^{(i)}. \quad (68)$$

In contrast to the static, non-rotating Schwarzschild-type isolated horizon model (Engle et al., 2010), which directly fulfills Eq. (68), the pullback of the Ashtekar-Barbero curvature to the S^2 -sections, given by Eq. (66), yields a boundary constraint, which has a non-constant factor and merely depends on the first component of the projected Plebanski 2-form

$$\mathbf{F}_\gamma^{(i)} = c_0^{(i)}(\theta) \Sigma^{(1)}. \quad (69)$$

Thus, the available structure does not satisfy the imposed condition. Hence, the curvature has to be appropriately transformed to match the form of Eq. (68). Following the procedure explained in FIG. 2, one can define a new Ashtekar-Barbero curvature \mathcal{A}_γ , using the original curvature \mathbf{A}_γ , via

$$\mathcal{A}_\gamma = \varphi^*(G\mathbf{A}_\gamma G^{-1} - d(G)G^{-1}), \quad (70)$$

where $G \in \text{SU}(2)$ and φ^* is the pullback of an active diffeomorphism, with a corresponding curvature $\mathcal{F}_\gamma = \mathcal{F}(\mathcal{A}_\gamma)$ that leads to the required result by construction. A quantum version of this constraint, together with a conserved presymplectic structure, to capture the degrees of freedom of the stationary, axially symmetric isolated horizon boundary in terms of a $\text{SU}(2)$ Chern-Simons theory, can then be implemented within the loop quantum gravity framework.

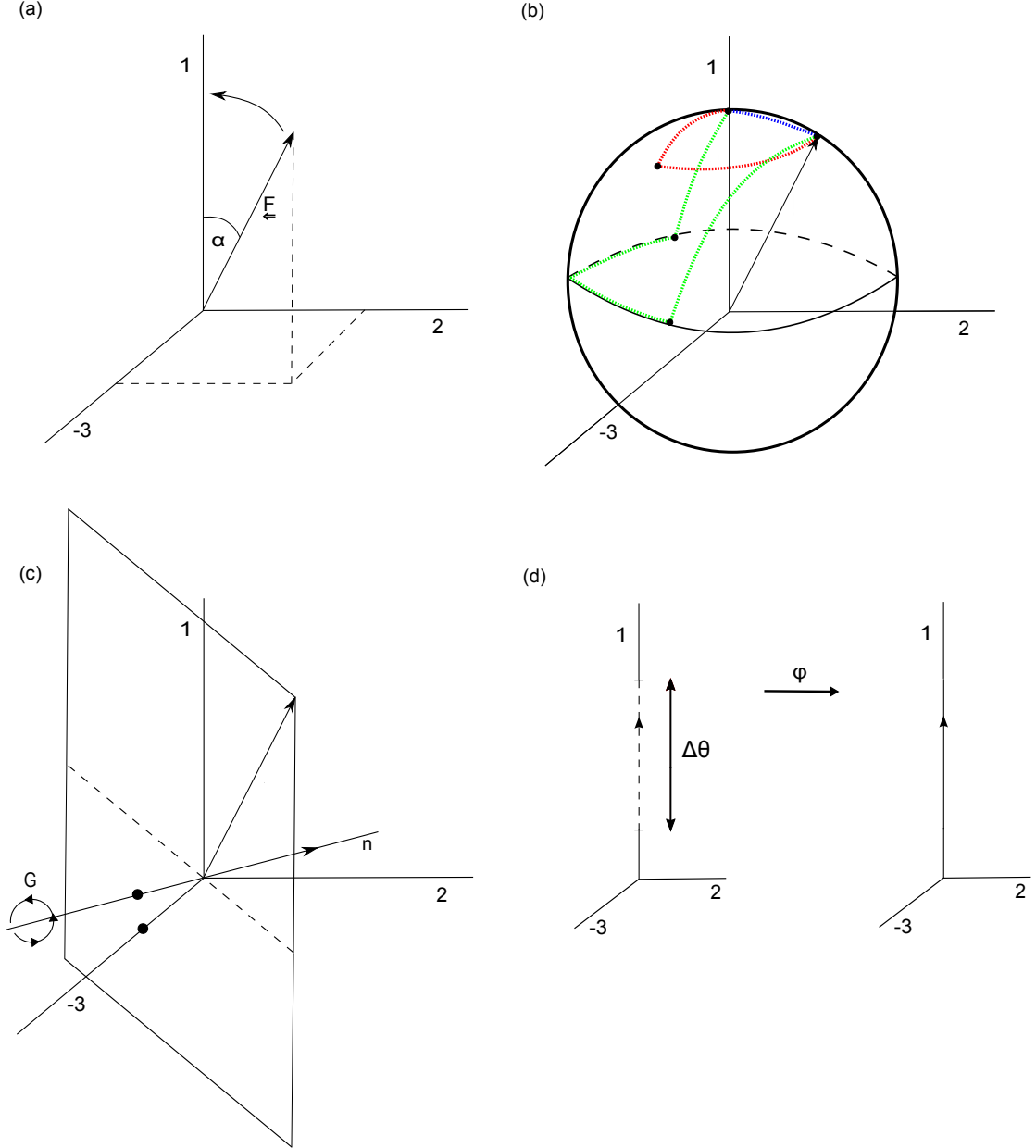


FIG. 2: Procedure to construct a curvature constraint for rotating Kerr isolated horizon boundaries in the form $\underline{\underline{F}}_\gamma = \text{const.} \times \underline{\underline{\Sigma}}$, which is a key ingredient for models of quantum isolated horizons in the framework of loop quantum gravity. Starting with a curvature, as given in Eq. (66), one can consider it as merely a vector in \mathbb{R}^3 (see (a)), and align it with the 1-direction by acting on it with a simple $\text{SO}(3)$ -rotation G around an angle α in order to obtain a curvature equation $\underline{\underline{F}}_\gamma = c_1(\theta)\underline{\underline{\Sigma}}$, where $c_1(\theta)$ is a function depending solely on the azimuthal angle θ . Note that in the particular gauge used in this study, the pullback of the Plebanski 2-form has just one non-vanishing component (in the 1-direction), and, therefore, the rotation is constructed to fit this condition by directly yielding a curvature with $\underline{\underline{F}}_\gamma^{(2)} = \underline{\underline{F}}_\gamma^{(3)} = 0$. The method of implementing the rotation is ambiguous as shown in (b). Here, one takes the most natural choice, fulfilling a minimum principle, i.e., one chooses a rotation that transports $\underline{\underline{F}}$ along a geodesic on the enclosing sphere (blue line). This specific geodesic $\text{SO}(3)$ -transformation can be computed by rotating the curvature around a normal of the 1- $\underline{\underline{F}}$ plane as illustrated in (c). Subsequently, as described in (d), applying an active diffeomorphism φ to the aligned curvature results in a deformation of the function $c_1(\theta)$, such that it becomes a constant.

VI. CONCLUSIONS

Both classical and quantum descriptions of rotating isolated horizon geometries are of major relevance not only in theoretical and mathematical physics, but especially in all fields that are concerned with any sort of black hole dynamics like astrophysics, cosmology, numerical relativity and quantum gravity. This comes from the necessity of having a physically meaningful description of black holes, which is local in space and time. Isolated horizons are exactly the kind of structures that meet these requirements. Therefore, it is imperative to have a proper, transparent characterization of them available. In this study, the complex, self-dual Ashtekar and the real Ashtekar-Barbero boundary connection variable formulations of classical Kerr isolated horizons, for a Hamiltonian representation of rotating black holes, were derived. The presented generalization of the already extensively investigated static, non-rotating Schwarzschild-type isolated horizons to stationary, axisymmetric geometries, with non-zero, non-trivial charge $a = J/M$, generating rigid rotations around a symmetry axis, has not been achieved before.

The $SL(2, \mathbb{C})$ and the $SU(2)$ Kerr isolated horizon boundary connections and curvatures, in which the degrees of freedom of type II Kerr isolated horizons are encoded, have been constructed in an instructive, straightforward way, by starting from a generic Kerr solution of tetrad gravity in a well-defined coordinate system at the outer event horizon at $r = r_+$, which was realized via an analytic Kruskal-like extension. The first Maurer-Cartan structure equation was solved in order to obtain the spin connection. Afterward, isolated horizon boundary conditions, the time gauge fixing, and external bulk structure information were imposed on the Newman-Penrose auxiliary fields and on the pullbacks of the spatiotemporal tetrad and the spin connection to the S^2 -sections of the horizon foliation. Finally, having derived the proper tetrad frame and spin connection, which capture the degrees of freedom of rotating Kerr isolated horizon geometries, the complex, self-dual Ashtekar and the real Ashtekar-Barbero connections and their corresponding curvatures were computed, respectively. The latter Ashtekar-Barbero curvature boundary constraint equation is vital for the canonical quantization of Kerr isolated horizon geometries in the framework of loop quantum gravity.

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Appendix A: Proof of Isolated Horizon Boundary Conditions (1) and (6) in the Newman-Penrose Formalism

By direct calculation, it can be shown that the real Newman-Penrose 1-form l'' , which constitutes the equivalence class of null normals $[l'']_{\sim}$ to the horizon Δ , has vanishing expansion $\theta_{(l'')} = 0$ and satisfies the commutator constraint $[\mathfrak{L}_{l''}, \mathfrak{D}] = 0$. The expansion scalar $\theta_{(l'')} = \underline{\overleftarrow{g}}_{\mu\nu} \nabla^{\mu} l''^{\nu}$ at Δ in the Newman-Penrose formalism, with l'' normal, m'' and m''^* tangent, and n'' transversal to Δ , reads (Ashtekar, Fairhurst & Krishnan, 2000)

$$\theta_{(l'')} \stackrel{\text{at } \Delta}{=} -2\Re\left(\underline{\overleftarrow{m}}_{\mu}'' \underline{\overleftarrow{m}}_{\nu}''^* \nabla^{\mu} l''^{\nu}\right). \quad (\text{A1})$$

Expressing $m_{\nu}''^*$ and l''^{ν} in terms of the spatiotemporal basis (47), and applying the identity $\tilde{\omega}^{(a)(b)\mu} = \tilde{e}^{(a)}_{\nu} \nabla^{\mu} \tilde{e}^{(b)\nu}$, one obtains

$$\theta_{(l'')} = -\Re\left(\underline{\overleftarrow{m}}_{\mu}'' \left(\tilde{e}^{(2)}_{\nu} + i\tilde{e}^{(3)}_{\nu}\right) \nabla^{\mu} \left(\tilde{e}^{(0)\nu} - \tilde{e}^{(1)\nu}\right)\right) = -\Re\left(\underline{\overleftarrow{m}}_{\mu}'' \left[\tilde{\omega}^{(2)(0)\mu} - \tilde{\omega}^{(2)(1)\mu} + i\left(\tilde{\omega}^{(3)(0)\mu} - \tilde{\omega}^{(3)(1)\mu}\right)\right]\right). \quad (\text{A2})$$

Since the connection equalities $\underline{\overleftarrow{\omega}}^{(0)(2)} = \underline{\overleftarrow{\omega}}^{(1)(2)}$ and $\underline{\overleftarrow{\omega}}^{(0)(3)} = \underline{\overleftarrow{\omega}}^{(1)(3)}$ hold at the horizon, which can be derived from Eqs. (27), (30), (33), and (48), the differences in Eq. (A2) vanish identically, immediately resulting in $\theta_{(l'')} = 0$. This proves that all elements of the equivalence class $[l'']_{\sim}$ are expansion-free at Δ .

In order to verify that the commutator $[\mathfrak{L}_{l''}, \mathfrak{D}]$ vanishes at the horizon, when operating on forms $V \in T^*\Delta$ tangent and forms $W \in [l'']_{\sim}$ normal to Δ , it is sufficient to examine its action on each of the Newman-Penrose 1-forms specified in Eq. (43). For this purpose, one defines the Lie derivative \mathfrak{L} of differential forms Y , which is related to the exterior derivative d and the interior product \lrcorner

$$(U \lrcorner Y)_{\mu_1 \dots \mu_{p-1}} = U^{\nu} Y_{\nu \mu_1 \dots \mu_{p-1}}, \quad (\text{A3})$$

where $U \in TM$ and $(Y_{\mu_1 \dots \mu_p}) \in \Lambda^p$ denote a vector field and a p -form, respectively, by Cartan's identity

$$\mathfrak{L}_U Y = d(U \lrcorner Y) + U \lrcorner dY. \quad (\text{A4})$$

Then, applying the commutator $[\mathfrak{L}_{l''}, \mathfrak{D}]$ to \mathbf{n}'' , gives

$$\begin{aligned} [\mathfrak{L}_{l''}, \mathfrak{D}] \mathbf{n}'' &\stackrel{\text{at } \Delta}{=} \underbrace{\mathfrak{L}_{l''}(\nabla \mathbf{n}'')}_{\leftarrow} - \underbrace{\nabla(\mathfrak{L}_{l''} \mathbf{n}'')}_{\leftarrow} = \frac{1}{\sqrt{2}} \underbrace{\mathfrak{L}_{l''}(\nabla_\mu \tilde{e}^{(0)}_\nu + \nabla_\mu \tilde{e}^{(1)}_\nu) dx^\mu \otimes dx^\nu}_{\leftarrow} - \underbrace{\nabla(d(\mathbf{l}'' \lrcorner \mathbf{n}'') + \mathbf{l}'' \lrcorner d\mathbf{n}'')}_{\leftarrow} \\ &= \frac{1}{\sqrt{2}} \underbrace{\mathfrak{L}_{l''}(\tilde{\omega}^{(a)(0)}_\mu + \tilde{\omega}^{(a)(1)}_\mu) \tilde{e}_{(a)\nu} dx^\mu \otimes dx^\nu}_{\leftarrow}. \end{aligned} \quad (\text{A5})$$

Substituting (43) and (44) in the last term in line 1 of (A5) shows that this contribution vanishes. Expanding the remaining term in line 2 according to the Leibniz rule, yields

$$\begin{aligned} [\mathfrak{L}_{l''}, \mathfrak{D}] \mathbf{n}'' &\stackrel{\text{at } \Delta}{=} \underbrace{\mathfrak{L}_{l''}(-\tilde{\omega}^{(0)(1)} \otimes \mathbf{l}'' + 2\Re[(\tilde{\omega}^{(0)(3)} + i\tilde{\omega}^{(0)(2)}) \otimes \mathbf{m}''])}_{\leftarrow} = \underbrace{-\tilde{\omega}^{(0)(1)} \otimes (\mathfrak{L}_{l''} \mathbf{l}'')}_{\leftarrow} - \underbrace{(\mathfrak{L}_{l''} \tilde{\omega}^{(0)(1)}) \otimes \mathbf{l}''}_{\leftarrow} \\ &\quad + 2\Re \left[\underbrace{(\tilde{\omega}^{(0)(3)} + i\tilde{\omega}^{(0)(2)}) \otimes (\mathfrak{L}_{l''} \mathbf{m}'')}_{\leftarrow} + \underbrace{(\mathfrak{L}_{l''}(\tilde{\omega}^{(0)(3)} + i\tilde{\omega}^{(0)(2)})) \otimes \mathbf{m}''}_{\leftarrow} \right]. \end{aligned} \quad (\text{A6})$$

With $\tilde{\omega}^{(0)(1)} = \tilde{\omega}^{(0)(1)}_\theta(\theta) d\theta + \tilde{\omega}^{(0)(1)}_{\tilde{\varphi}_+}(\theta) d\tilde{\varphi}_+$, $\tilde{\omega}^{(0)(2)} = \tilde{\omega}^{(0)(2)}_\theta(\theta) d\theta + \tilde{\omega}^{(0)(2)}_{\tilde{\varphi}_+}(\theta) d\tilde{\varphi}_+ + \lambda_1(\theta)/L dL$, and $\tilde{\omega}^{(0)(3)} = \tilde{\omega}^{(0)(3)}_\theta(\theta) d\theta + \tilde{\omega}^{(0)(3)}_{\tilde{\varphi}_+}(\theta) d\tilde{\varphi}_+ + \lambda_2(\theta)/L dL$, where $\lambda_{1/2}(\theta)$ denote real-valued functions that depend on the azimuthal angle θ , one can immediately compute the pullbacks of the Lie derivatives of the spin connection components as $\mathfrak{L}_{l''} \tilde{\omega}^{(0)(1)} = \mathfrak{L}_{l''} \tilde{\omega}^{(0)(2)} = \mathfrak{L}_{l''} \tilde{\omega}^{(0)(3)} = 0$. Again using (43) and (44), one also directly finds $\mathfrak{L}_{l''} \mathbf{l}'' = \mathfrak{L}_{l''} \mathbf{m}'' = 0$, leading to $[\mathfrak{L}_{l''}, \mathfrak{D}] \mathbf{n}'' = 0$ at the horizon. Based on the same methodology, one can prove that for the other Newman-Penrose 1-forms the same result holds.

Appendix B: Slowly Rotating Kerr Isolated Horizons in the SU(2) Ashtekar-Barbero Formalism

In the limit of small angular momentum $a/r_+ \ll 1$, one can take the first-order a/r_+ -approximation of the Ashtekar-Barbero connection (64) and the curvature (66), respectively, leading to the following expressions

$$\begin{aligned} \bar{A}^{(1)}_\gamma &= -\frac{a \sin(\theta)}{r_+} d\theta + \left[\cos(\theta) + \frac{3\gamma a \sin^2(\theta)}{2r_+} \right] d\tilde{\varphi}_+ \\ \bar{A}^{(2)}_\gamma &= \frac{1}{\sqrt{2}r_+} \left[(\gamma r_+ + a \cos(\theta)) d\theta - \sin(\theta) (r_+ - \gamma a \cos(\theta)) d\tilde{\varphi}_+ \right] \\ \bar{A}^{(3)}_\gamma &= \frac{1}{\sqrt{2}r_+} \left[(r_+ - \gamma a \cos(\theta)) d\theta + \sin(\theta) (\gamma r_+ + a \cos(\theta)) d\tilde{\varphi}_+ \right] \end{aligned} \quad (\text{B1})$$

for the slowly rotating Kerr isolated horizon Ashtekar-Barbero connection components and

$$\begin{aligned} \bar{F}^{(1)}_\gamma &= \frac{1}{r_+^2} \left[\frac{\gamma^2 - 1}{2} + \frac{3\gamma a \cos(\theta)}{r_+} \right] \bar{\Sigma}^{(1)} \\ \bar{F}^{(2)}_\gamma &= -\frac{1}{\gamma} \bar{F}^{(3)}_\gamma = \frac{3\gamma a \sin(\theta)}{2\sqrt{2}r_+^3} \bar{\Sigma}^{(1)} \end{aligned} \quad (\text{B2})$$

for the corresponding curvature components. The simplicity of these first-order formulas, especially the emerging proportionality between $\bar{F}^{(2)}_\gamma$ and $\bar{F}^{(3)}_\gamma$ in (B2), gives rise to a reasonably transparent study of slowly rotating Kerr isolated horizons in both classical and quantum regimes. However, recent observations in astrophysics (McClintock et al., 2006) show that most of the black holes actually appear to rotate quite rapidly, and, thus, probably constitute the most common form existing in nature. This motivates to rather study the more general curvature Eq. (66). Nonetheless, it is also interesting to look at the case of slow rotation, which can result in partial insights into the full theory.

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